CHALMERS
UNIVERSITY OF TECHNOLOGY

Automorphic string amplitudes

Henrik Gustafsson

Séminaire de matrices, cordes et géométries aléatoires

Institut de Physique Théorique, Saclay 2016

hgustafsson.se

Based on

Small automorphic representations and degenerate Whittaker vectors
HG, Axel Kleinschmidt, Daniel Persson
`arXiv:1412.5625 [math.NT]` [GKP14]
Journal of Number Theory 166 (Sep, 2016) 344–399

Eisenstein series and automorphic representations
Philipp Fleig, HG, Axel Kleinschmidt, Daniel Persson
`arXiv:1511.04265 [math.NT]` [FGKP15]
Cambridge University Press (2017)

Upcoming work with
Olof Ahlén, Dmitry Gourevitch, AK, Baiying Liu, DP, Siddhartha Sahi

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$SL(n)$ E_6, E_7, E_8

Outline

Compute Fourier coefficients of automorphic forms to capture information about non-perturbative effects such as instantons and black holes

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- Scattering amplitudes
  4-graviton | Derivative expansion | U-duality | SUSY constraints

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- **Scattering amplitudes**
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- **Fourier coefficients**
  Parabolic subgroups | Limits of string theory | Adelic framework

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- Outlook

Motivation

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- Hecke eigenvalues
- Point counts of elliptic curves
- Langlands program
  L-functions | The Langlands–Shahidi method

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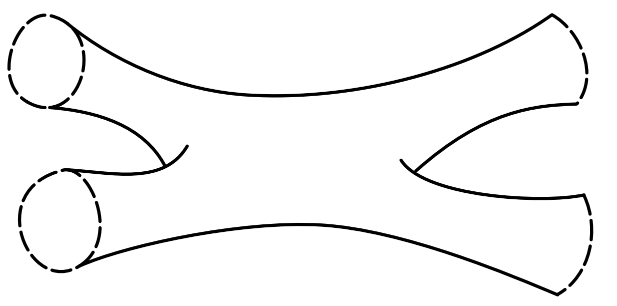
String theory

Toroidal compactifications of type IIB string theory

4-graviton scattering amplitudes



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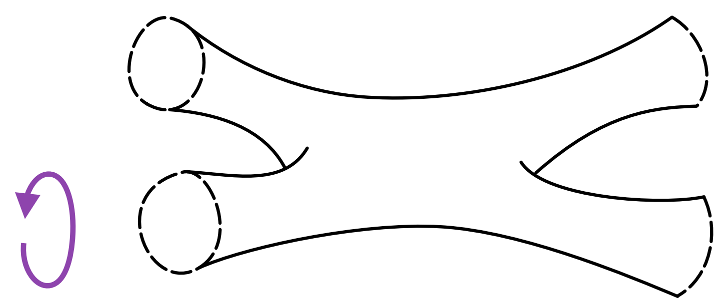
$\alpha' = \ell_{\mathrm{s}}^2$



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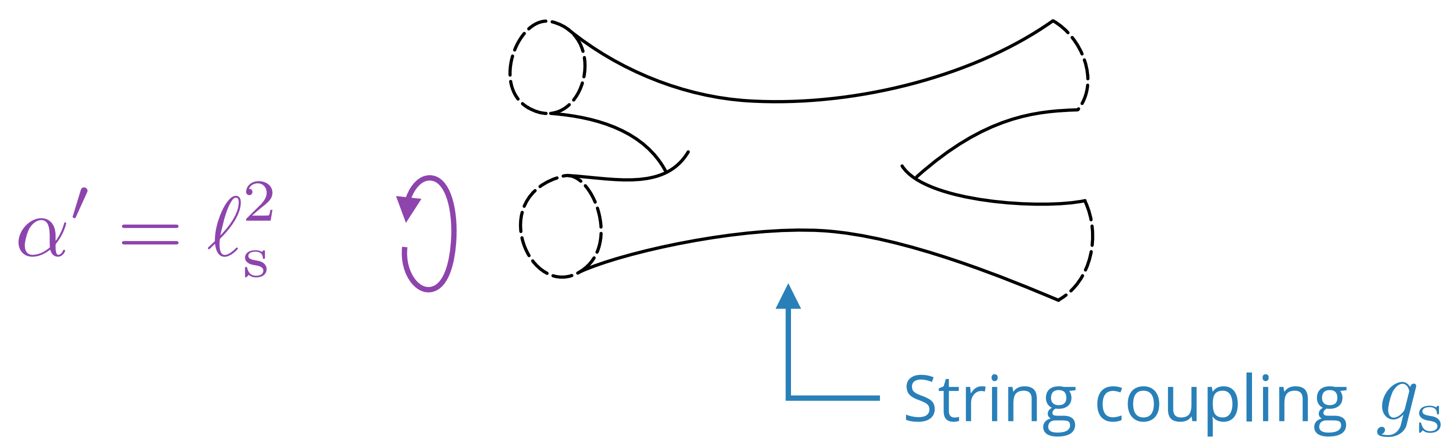

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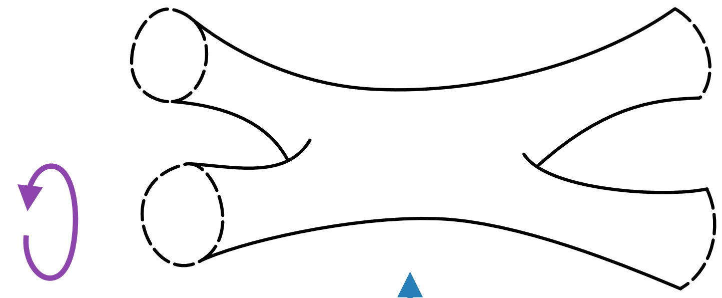
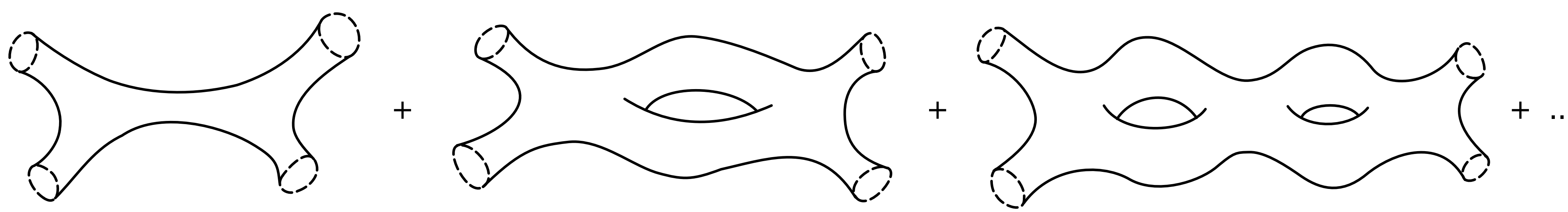


String coupling g_s

$$s = -\frac{\alpha'}{4}(k_1 + k_2)^2 \qquad t = -\frac{\alpha'}{4}(k_1 + k_3)^2 \qquad u = -\frac{\alpha'}{4}(k_1 + k_4)^2$$

Interactions

g_s^{-2} $\qquad$ 1 $\qquad$ g_s^2



4-graviton amplitude in 10 dimensions:

[Green-Schwarz, Green-Schwarz-Brink, Gross-Witten]

Interactions

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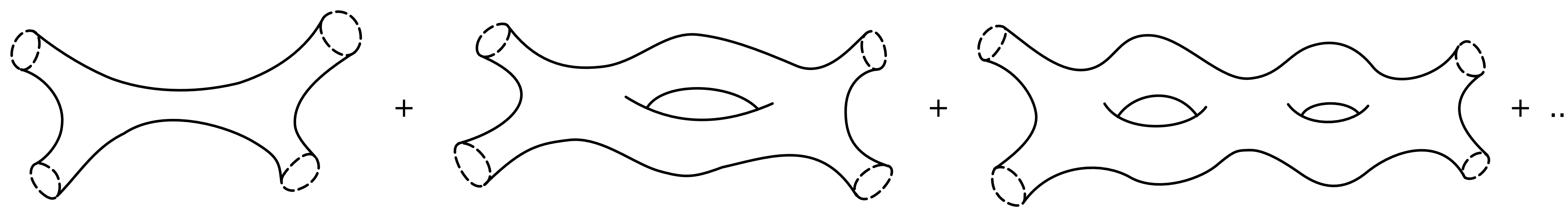

4-graviton amplitude in 10 dimensions:

$$\mathcal{A} = \left(g_s^{-2} \frac{1}{stu} \frac{\Gamma(1-s)\Gamma(1-t)\Gamma(1-u)}{\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)} \right) \mathcal{R}^4$$

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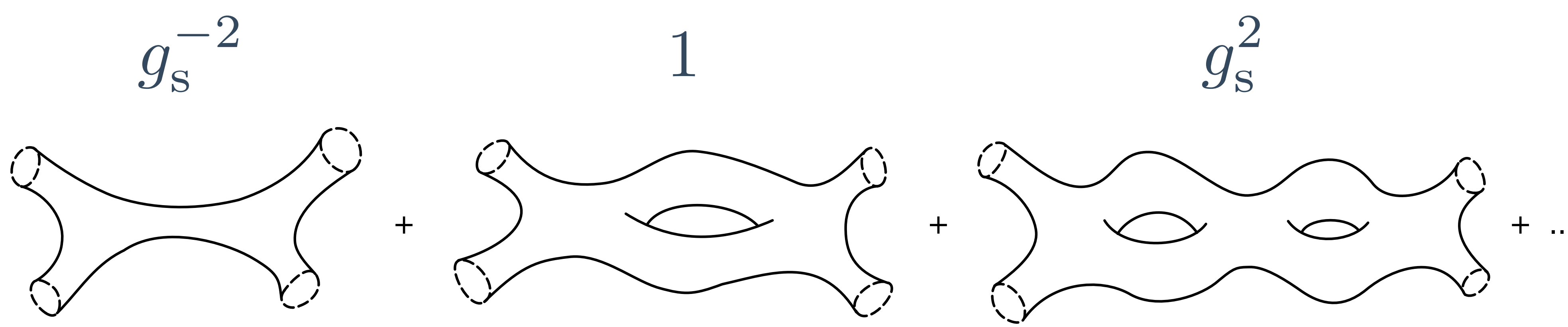

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Kinematic term:
Contraction of 4 linearized Riemann tensors
$t_8 t_8 \mathcal{R}^4$

[Green-Schwarz, Green-Schwarz-Brink, Gross-Witten]

Interactions

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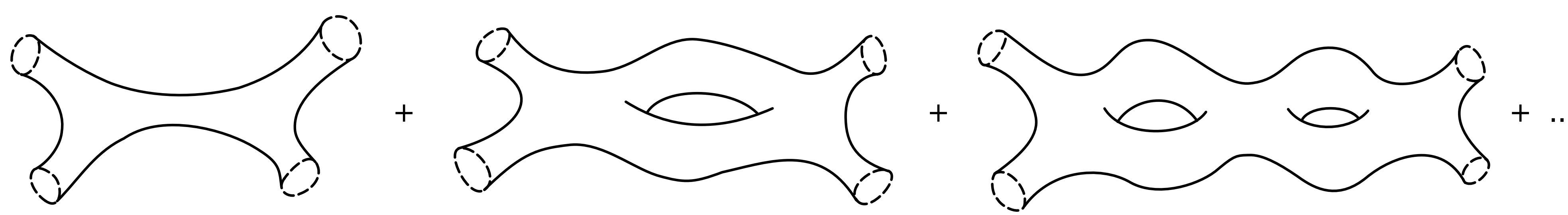

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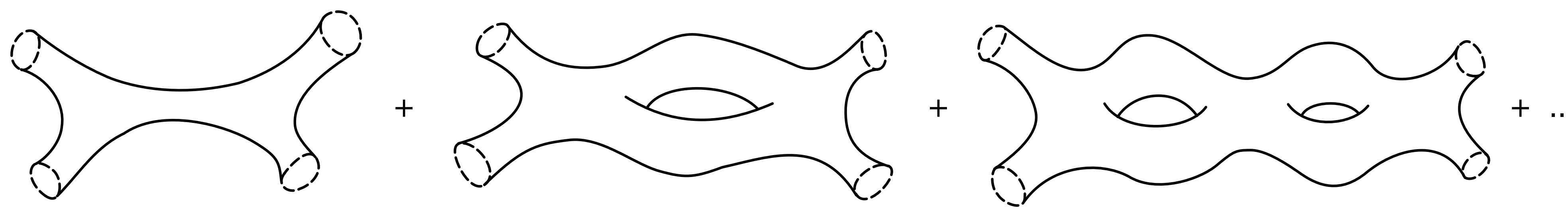

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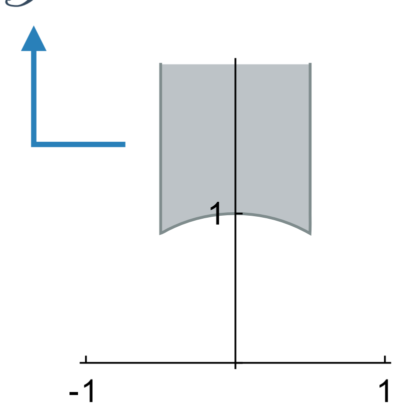

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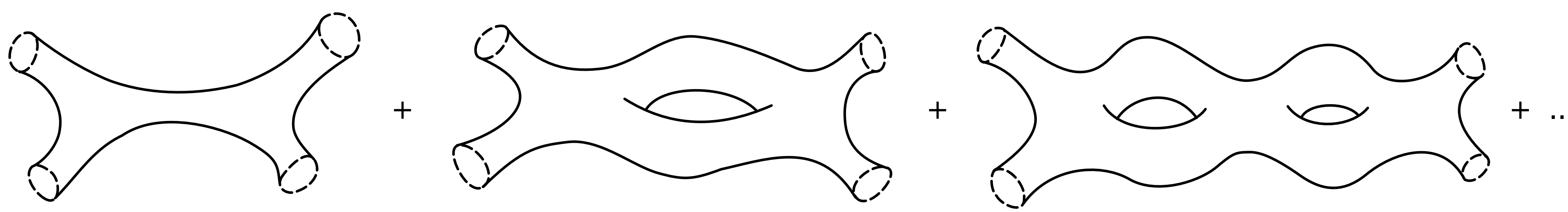

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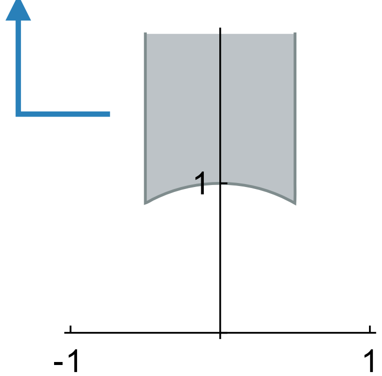
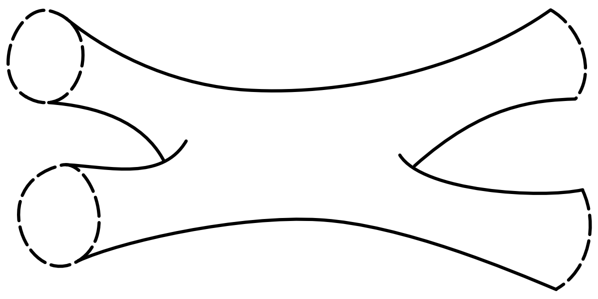


Modular invariant function

[Green-Schwarz, Green-Schwarz-Brink, Gross-Witten]

Interactions

String theory



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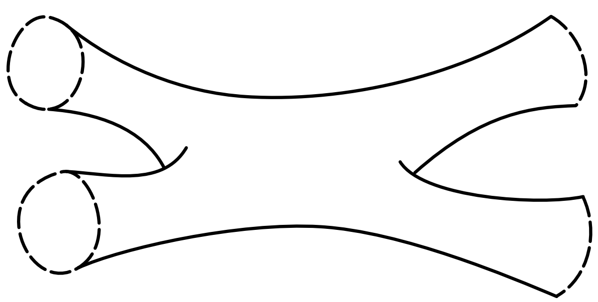
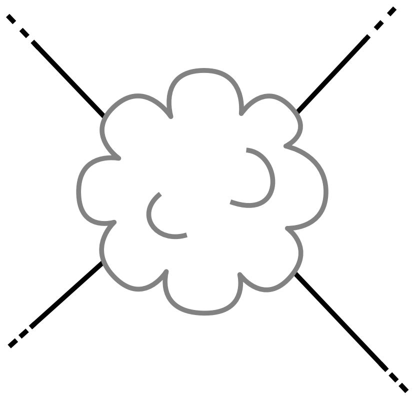


Effective field theory

Supergravity + $\mathcal{O}(\alpha')$



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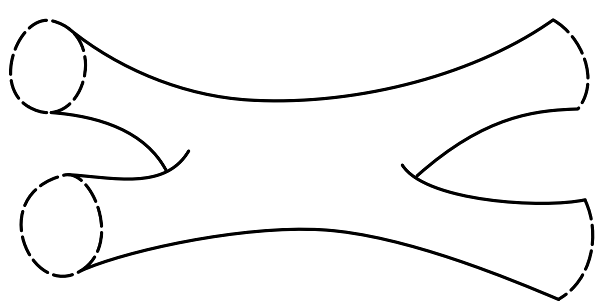
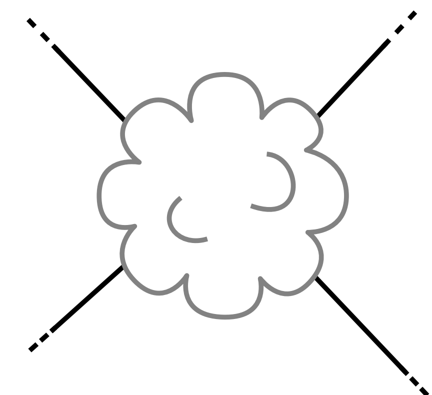


Effective field theory $\longrightarrow$

Supergravity + $\mathcal{O}(\alpha')$

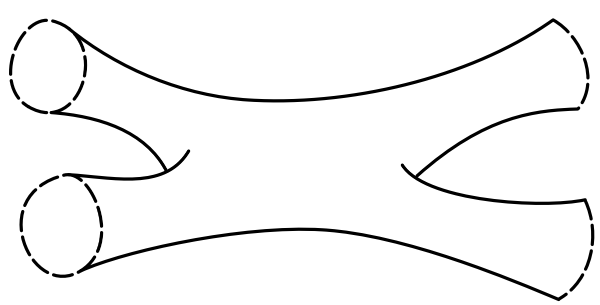


$s, t, u = \mathcal{O}(\alpha')$ $\quad p \longleftrightarrow \partial \implies \alpha'$ -expansion = ∂-expansion

Interactions

String theory

Supergravity + $\mathcal{O}(\alpha')$



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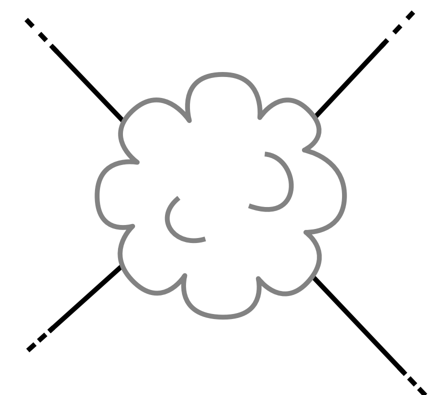

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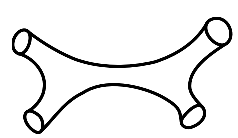

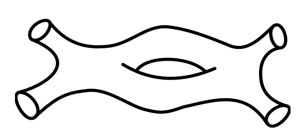

(Einstein frame)

$$\mathcal{L} \propto R + (\alpha')^3 \left(2\zeta(3) g_s^{-3/2} + 4\zeta(2) g_s^{1/2} + \ldots \right) R^4 + \ldots$$

Interactions

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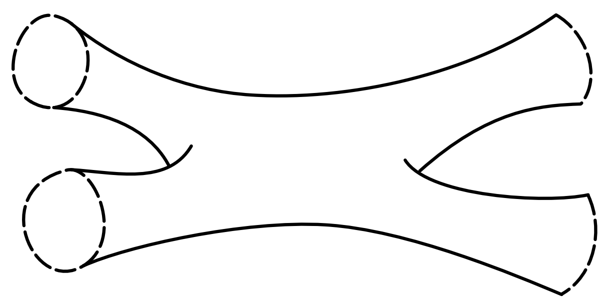

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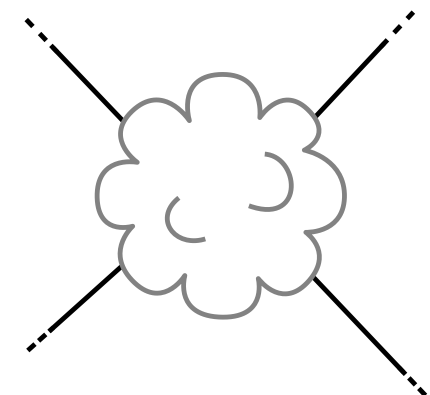

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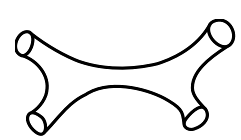

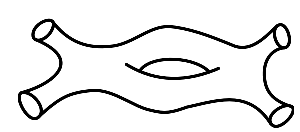

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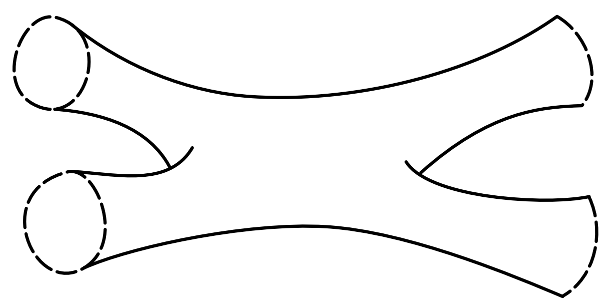
$$\mathcal{L} \propto R + (\alpha')^3 \Big(2\zeta(3) g_s^{-3/2} + 4\zeta(2) g_s^{1/2} + \dots\Big) R^4 + \dots$$

Contraction of 4 Riemann tensors

Interactions

String theory



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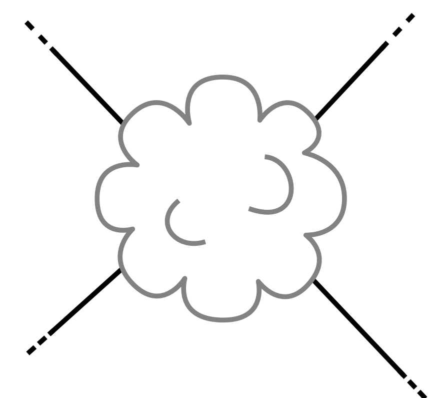

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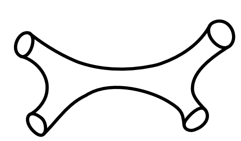

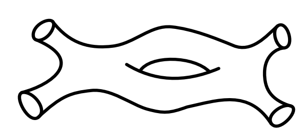

(Einstein frame)

$$
\begin{aligned}
\mathcal{L} \propto R + \; & (\alpha')^3 \Big(2\zeta(3) g_s^{-3/2} + 4\zeta(2) g_s^{1/2} + \ldots \Big) R^4 + \\
& (\alpha')^5 \Big(\zeta(5) g_s^{-5/2} + \ldots \Big) D^4 R^4 + \\
& (\alpha')^6 \Big(\tfrac{2}{3}\zeta(3)^2 g_s^{-3} + \tfrac{4}{3}\zeta(2)\zeta(3) g_s^{-1} + \ldots \Big) D^6 R^4 + \mathcal{O}((\alpha')^7)
\end{aligned}
$$

Interactions

$$\begin{aligned}\mathcal{L} \propto R + \ & (\alpha')^3 \Big(2\zeta(3) g_\mathrm{s}^{-3/2} + 4\zeta(2) g_\mathrm{s}^{1/2} + \ldots \Big) R^4 + \\ & (\alpha')^5 \Big(\zeta(5) g_\mathrm{s}^{-5/2} + \ldots \Big) D^4 R^4 + \\ & (\alpha')^6 \Big(\tfrac{2}{3}\zeta(3)^2 g_\mathrm{s}^{-3} + \tfrac{4}{3}\zeta(2)\zeta(3) g_\mathrm{s}^{-1} + \ldots \Big) D^6 R^4 + \mathcal{O}((\alpha')^7)\end{aligned}$$

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$$\begin{aligned}\mathcal{L} \propto R + \; & (\alpha')^3 \Big(2\zeta(3) g_{\mathrm{s}}^{-3/2} \; + \; 4\zeta(2) g_{\mathrm{s}}^{1/2} \qquad\qquad + \; \ldots \Big) \quad R^4 + \\ & (\alpha')^5 \Big(\zeta(5) g_{\mathrm{s}}^{-5/2} \qquad\qquad\qquad\qquad\qquad + \; \ldots \Big) D^4 R^4 + \\ & (\alpha')^6 \Big(\tfrac{2}{3}\zeta(3)^2 g_{\mathrm{s}}^{-3} \; + \; \tfrac{4}{3}\zeta(2)\zeta(3) g_{\mathrm{s}}^{-1} \; + \; \ldots \Big) D^6 R^4 + \mathcal{O}((\alpha')^7)\end{aligned}$$

$$\mathcal{L} \propto R + (\alpha')^3 \mathcal{E}_0(\tau) R^4 + (\alpha')^5 \mathcal{E}_4(\tau) D^4 R^4 + (\alpha')^6 \mathcal{E}_6(\tau) D^6 R^4 + \ldots$$

$$\tau = \chi + i g_{\mathrm{s}}^{-1}$$

Interactions

$$\begin{aligned}\mathcal{L} \propto R + \ &(\alpha')^3\Big(\underbrace{2\zeta(3)g_s^{-3/2} \ + \ 4\zeta(2)g_s^{1/2} \qquad\qquad + \ \ldots}\Big) \quad R^4 + \\ &(\alpha')^5\Big(\zeta(5)g_s^{-5/2} \qquad\qquad\qquad\qquad\qquad + \ \ldots\Big) D^4R^4 + \\ &(\alpha')^6\Big(\tfrac{2}{3}\zeta(3)^2g_s^{-3} \ + \ \tfrac{4}{3}\zeta(2)\zeta(3)g_s^{-1} \ + \ \ldots\Big) D^6R^4 + \mathcal{O}((\alpha')^7)\end{aligned}$$

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Interactions

$$\mathcal{L} \propto R + (\alpha')^3\left(2\zeta(3)g_s^{-3/2} + 4\zeta(2)g_s^{1/2} + \dots\right) R^4 +$$
$$(\alpha')^5\left(\zeta(5)g_s^{-5/2} + \dots\right) D^4R^4 +$$
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Interactions

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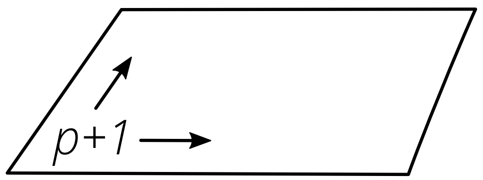
Bao, Basu, Bossard, Cederwall, Fleig, Green, Gubay, Gutperle, HG, Kazhdan, Kiritsis, Kleinschmidt, Lambert, Miller, Nilsson, Obers, Persson, Pioline, Russo, Sethi, Vanhove, Verschinin, Waldron, West, …

Non-perturbative effects

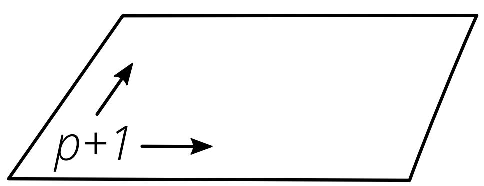
Non-perturbative effects



Dp-brane

p space directions
1 time direction

Non-perturbative effects



Dp-brane

p space directions
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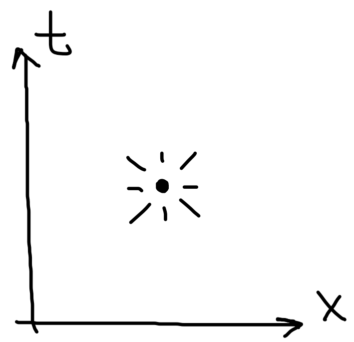
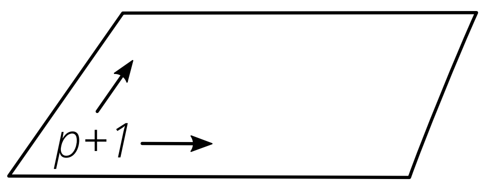


D-instanton

p = -1

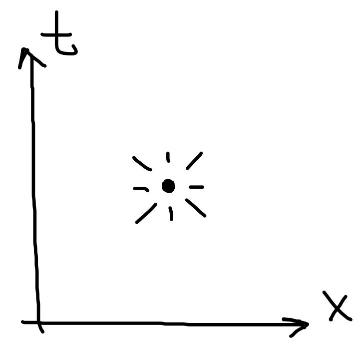
Non-perturbative effects



D*p*-brane

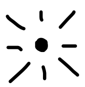
p space directions

1 time direction

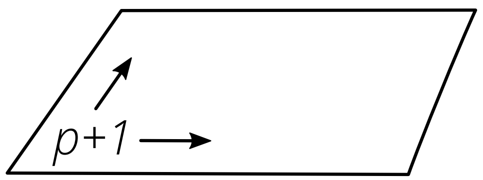


D-instanton

$p = -1$

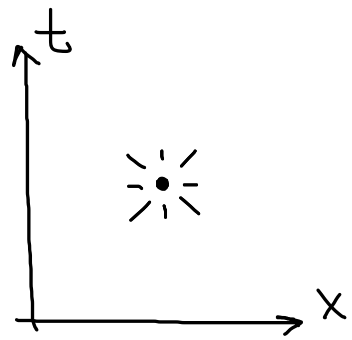


Non-perturbative effects



Dp-brane

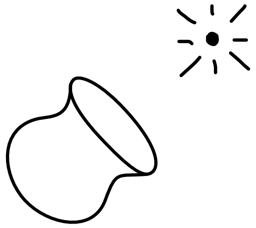
p space directions
1 time direction



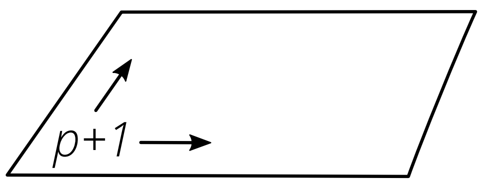
D-instanton

p = -1

$$\frac{1}{g_s}$$

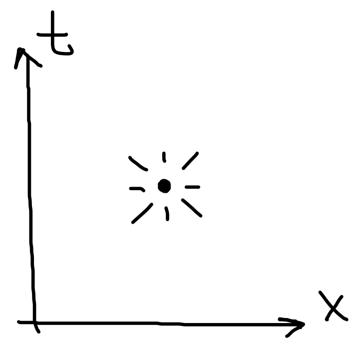


Non-perturbative effects



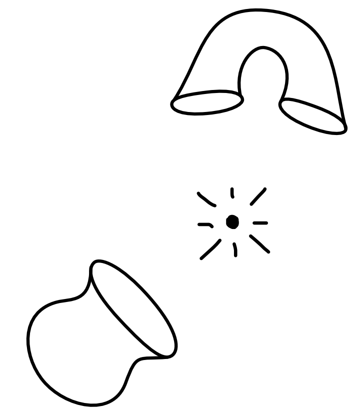
D*p*-brane

p space directions
1 time direction



D-instanton

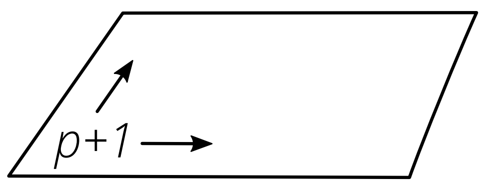
p = -1



1

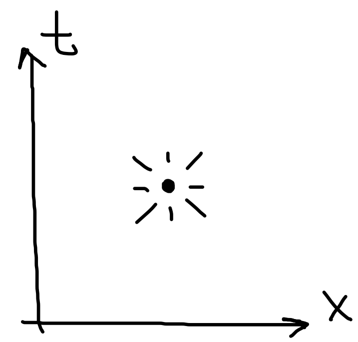
$\frac{1}{g_s}$

Non-perturbative effects



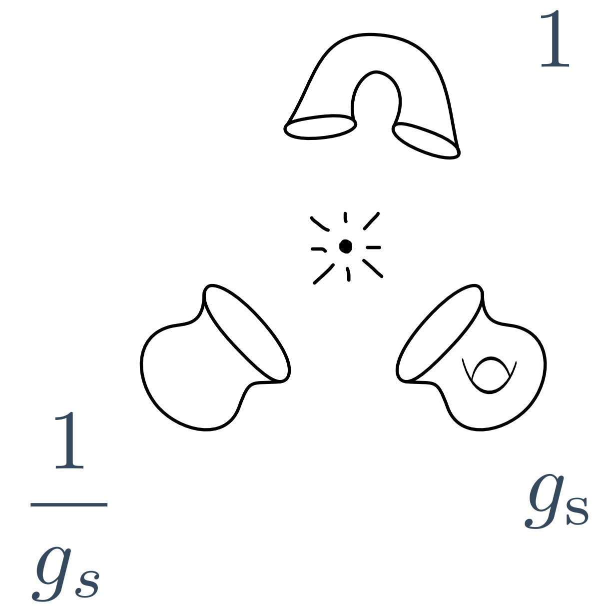
Dp-brane

p space directions
1 time direction

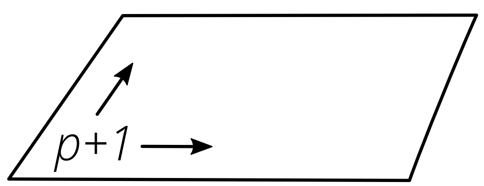


D-instanton

p = -1

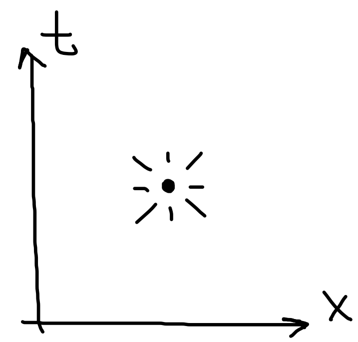


Non-perturbative effects



D*p*-brane

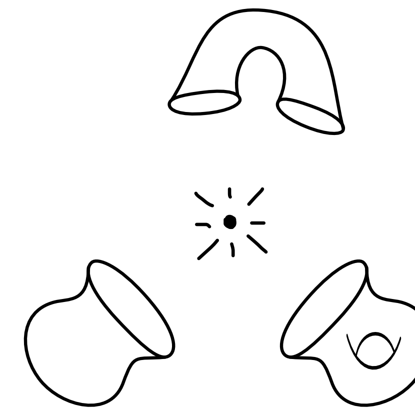
p space directions
1 time direction



D-instanton

p = -1

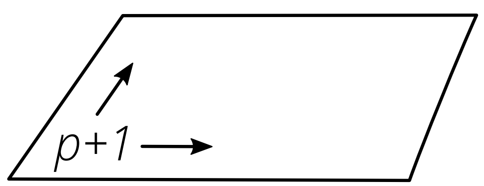
1



$\frac{1}{g_s}$ g_s

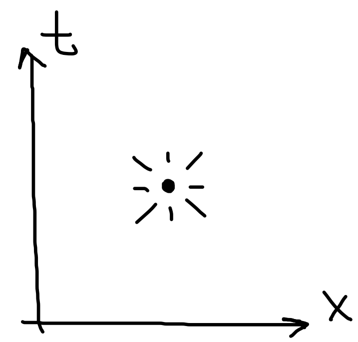
1

Non-perturbative effects



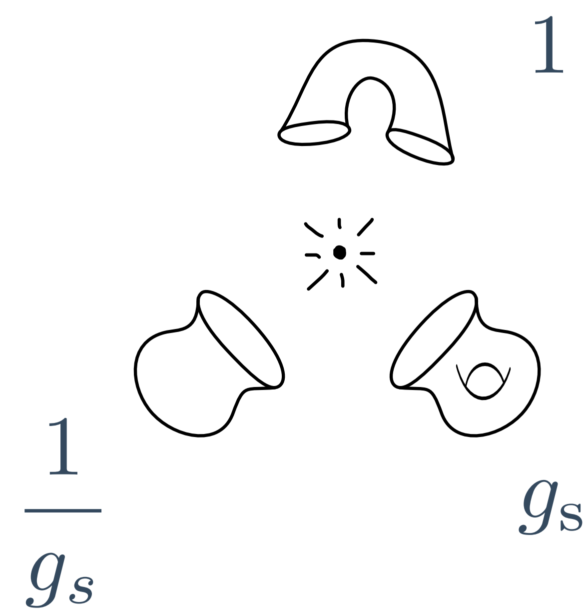
Dp-brane

p space directions
1 time direction

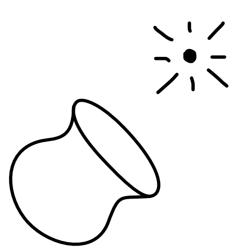


D-instanton

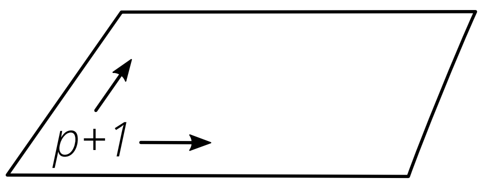
p = -1



$1 \quad +$

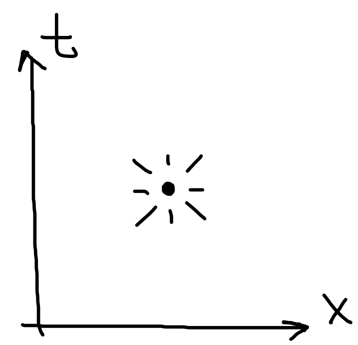


Non-perturbative effects



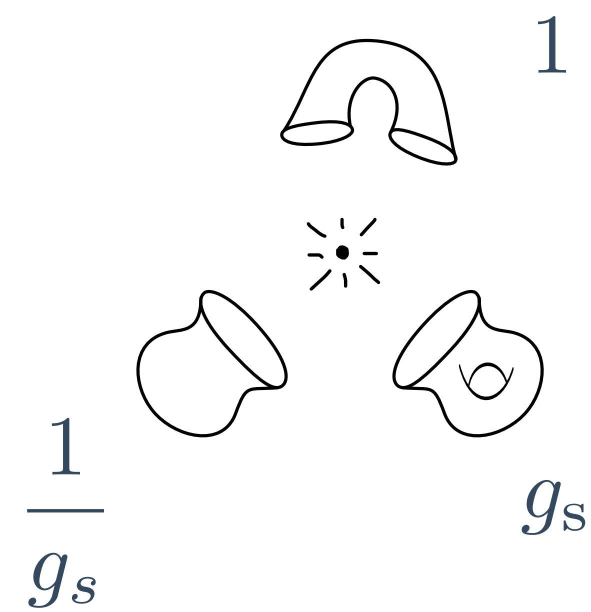
Dp-brane

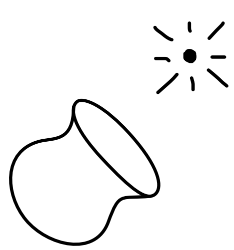
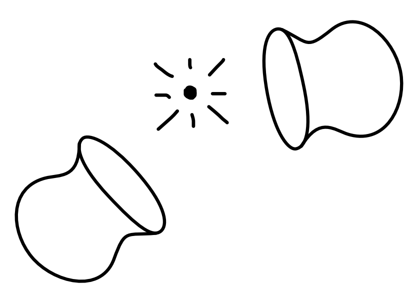
p space directions
1 time direction



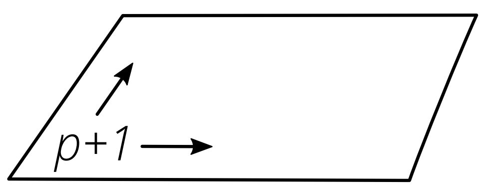
D-instanton

p = -1



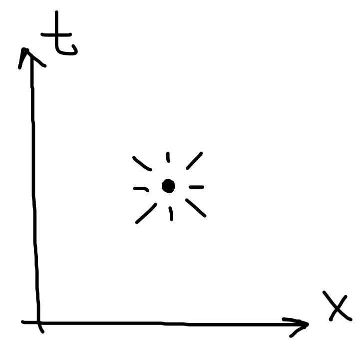
$1 \quad + $  $\quad + \quad \frac{1}{2!}$ 

Non-perturbative effects



D*p*-brane

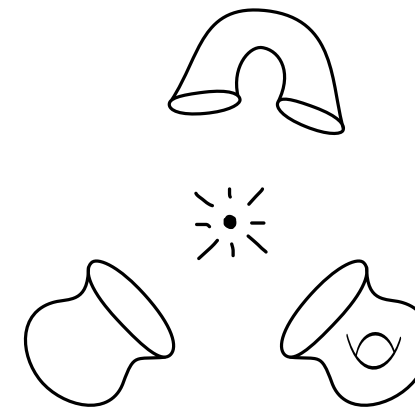
p space directions
1 time direction



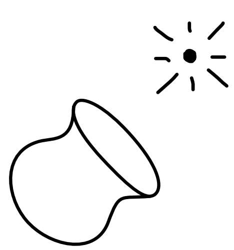
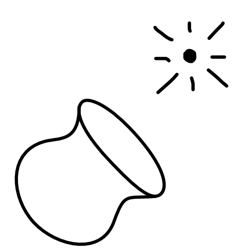
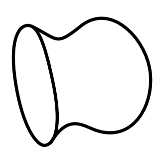
D-instanton

p = -1

1

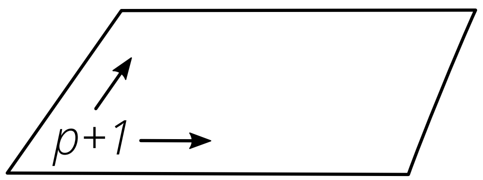


$\frac{1}{g_s}$ $\qquad$ g_s

$1 \quad + $  $\quad + \quad \frac{1}{2!}$  

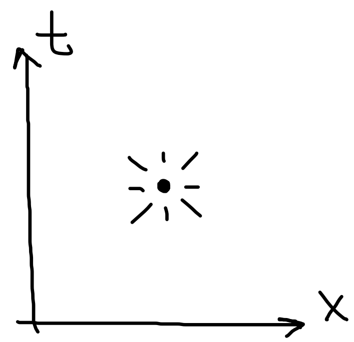
Symmetry factor for identical disks

Non-perturbative effects



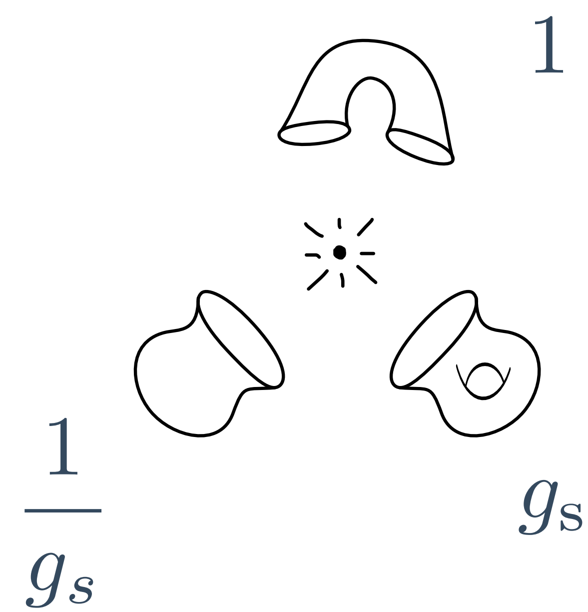
Dp-brane

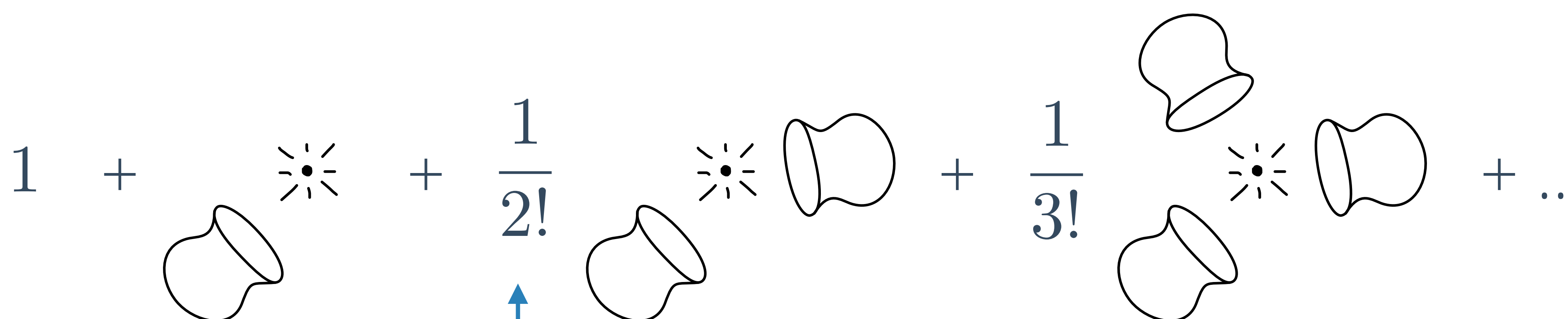
p space directions
1 time direction



D-instanton

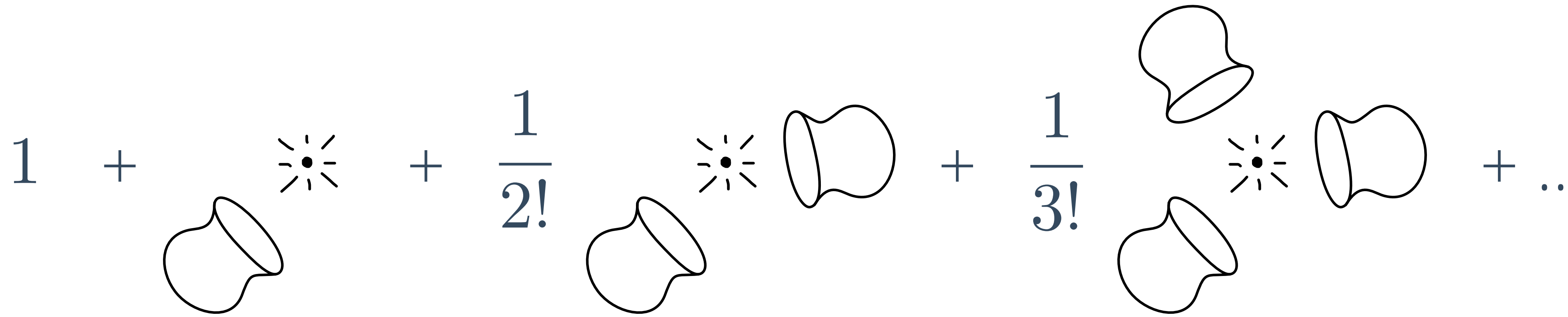
p = -1





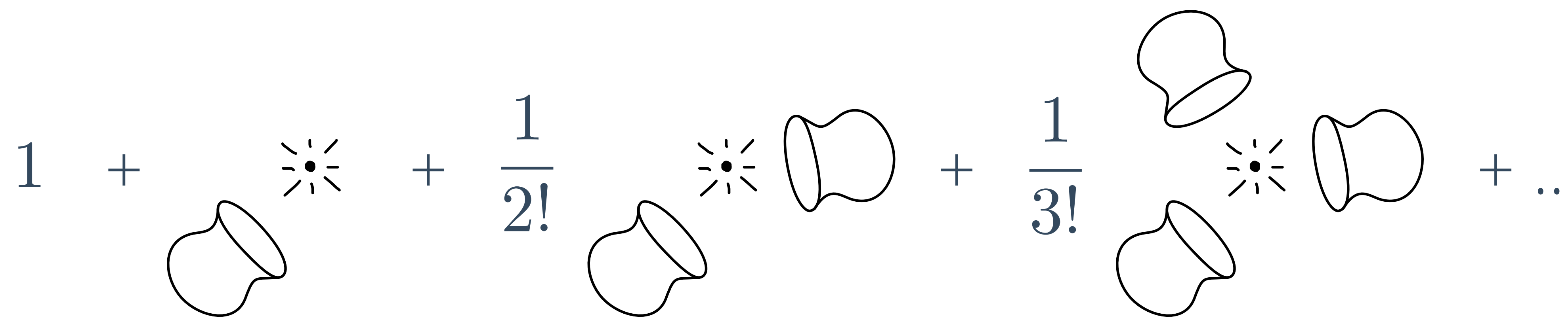
Symmetry factor for identical disks

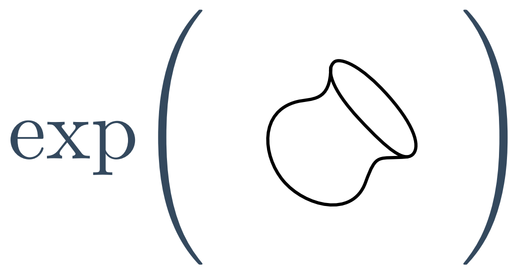
Non-perturbative effects



[Green-Gutperle]

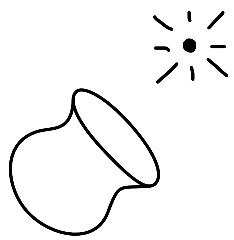
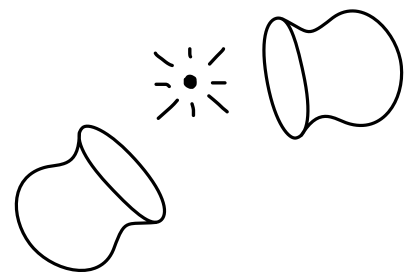
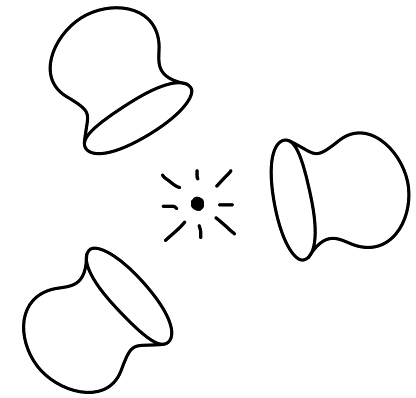
Non-perturbative effects

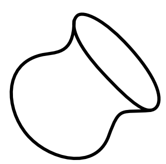




[Green-Gutperle]

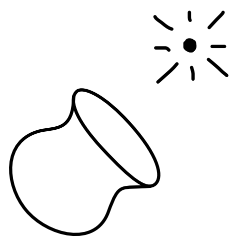
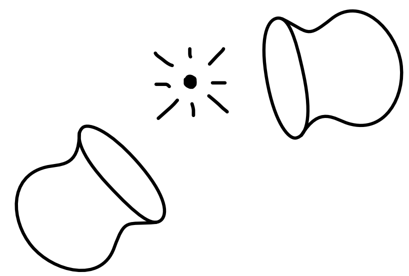
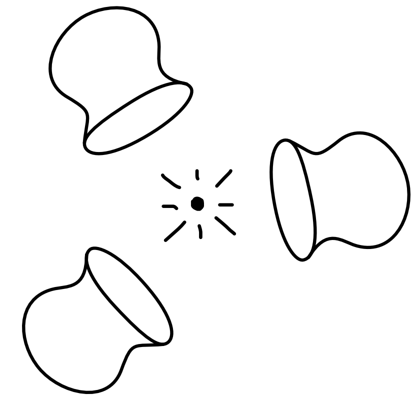
Non-perturbative effects

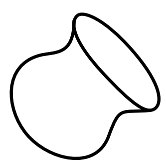
$1 \quad +$  $+ \quad \frac{1}{2!}$  $+ \quad \frac{1}{3!}$  $+ \ \ldots$

$\exp\Bigg($  $\Bigg) \sim \exp\left(-\frac{\text{const}}{g_{\text{s}}}\right)$

[Green-Gutperle]

Non-perturbative effects

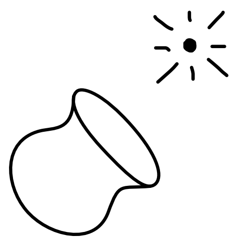
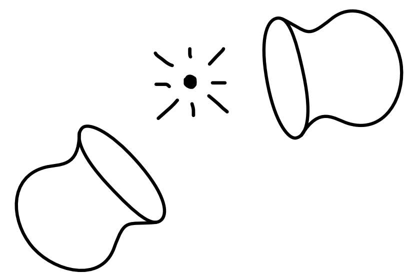
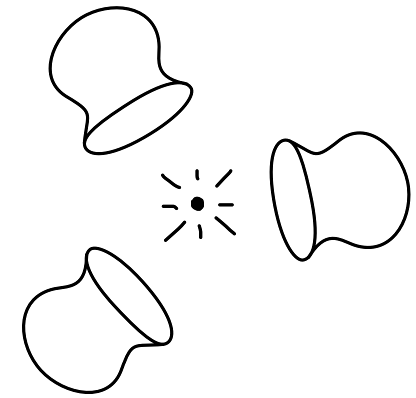
$1 \; + $  $+ \; \frac{1}{2!}$  $+ \; \frac{1}{3!}$  $+ \; \dots$

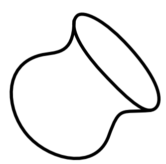
$\exp\Bigg($  $\Bigg) \;\sim\; \exp\left(-\frac{\mathrm{const}}{g_{\mathrm{s}}}\right)$

Non-perturbative in g_{s}

[Green-Gutperle]

Non-perturbative effects

$1 \;+$  $+\; \frac{1}{2!}$  $+\; \frac{1}{3!}$  $+\; \ldots$

$\exp\Big($  $\Big) \;\sim\; \exp\left(-\frac{\text{const}}{g_{\text{s}}}\right)$

Non-perturbative in g_{s}

$$\mathcal{E}_0(\tau) = 2\zeta(3)\tau_2^{3/2} + 4\zeta(2)\tau_2^{-1/2} + \ldots + Ce^{2\pi i\tau} + \ldots$$

$$\tau = \tau_1 + i\tau_2 = \chi + i g_{\text{s}}^{-1}$$

[Green-Gutperle]

Moduli space

$$R + (\alpha')^3 \mathcal{E}_0^{(D)}(g) R^4 + (\alpha')^5 \mathcal{E}_4^{(D)}(g) D^4 R^4 + (\alpha')^6 \mathcal{E}_6^{(D)}(g) D^6 R^4 + \ldots$$

$$\mathcal{M}_{\mathrm{classical}} = G(\mathbb{R})/K$$

[Cremmer-Julia]

Moduli space

$$R + (\alpha')^3 \mathcal{E}_0^{(D)}(g) R^4 + (\alpha')^5 \mathcal{E}_4^{(D)}(g) D^4 R^4 + (\alpha')^6 \mathcal{E}_6^{(D)}(g) D^6 R^4 + \ldots$$

$$\mathcal{M}_{\text{classical}} = G(\mathbb{R})/K$$

D	$G(\mathbb{R})$	K
10	$SL(2,\mathbb{R})$	$SO(2)$
9	$SL(2,\mathbb{R}) \times \mathbb{R}^+$	$SO(2)$
8	$SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$SO(3) \times SO(2)$
7	$SL(5,\mathbb{R})$	$SO(5)$
6	$Spin(5,5;\mathbb{R})$	$\big(Spin(5) \times Spin(5)\big)/\mathbb{Z}_2$
5	$E_6(\mathbb{R})$	$USp(8)/\mathbb{Z}_2$
4	$E_7(\mathbb{R})$	$SU(8)/\mathbb{Z}_2$
3	$E_8(\mathbb{R})$	$Spin(16)/\mathbb{Z}_2$

[Cremmer-Julia]

Moduli space

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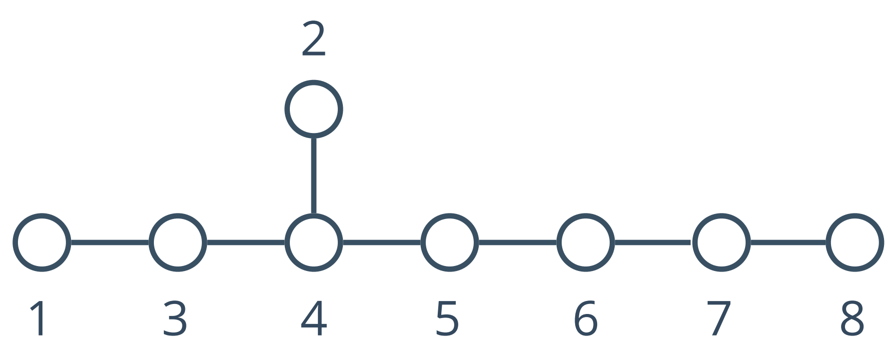

[Cremmer-Julia]

Moduli space

$$R+(\alpha')^3\mathcal{E}_0^{(D)}(g)R^4+(\alpha')^5\mathcal{E}_4^{(D)}(g)D^4R^4+(\alpha')^6\mathcal{E}_6^{(D)}(g)D^6R^4+\ldots$$

$$\mathcal{M}_{\text{classical}}=G(\mathbb{R})/K \qquad\qquad \mathcal{E}_n(\tau)=\mathcal{E}_n^{(10)}(g)$$

D	$G(\mathbb{R})$	K
10	$SL(2,\mathbb{R})$	$SO(2)$
9	$SL(2,\mathbb{R})\times\mathbb{R}^+$	$SO(2)$
8	$SL(3,\mathbb{R})\times SL(2,\mathbb{R})$	$SO(3)\times SO(2)$
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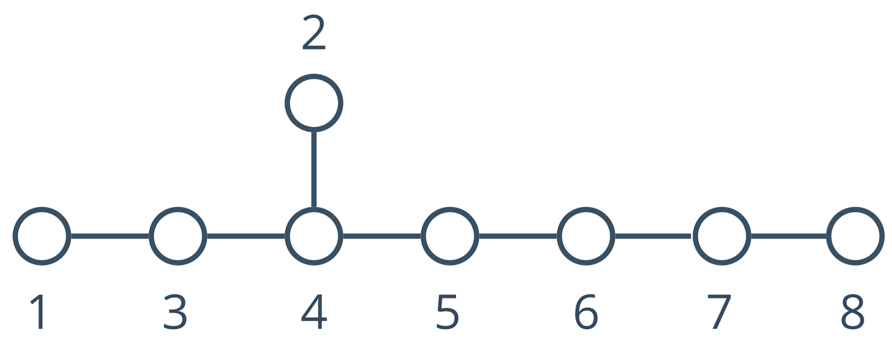

[Cremmer-Julia]

Moduli space

10 dimensions:

Moduli space

10 dimensions:

$$\tau = \chi + i g_{\mathrm{s}}^{-1} \in \mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\} \cong SL(2,\mathbb{R})/SO(2,\mathbb{R})$$

Moduli space

10 dimensions:

$$\tau = \chi + i g_{\mathrm{s}}^{-1} \in \mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\} \cong SL(2,\mathbb{R})/SO(2,\mathbb{R})$$

No similar structure for lower dimensions

U-duality

$G(\mathbb{R}) \circlearrowright \mathcal{M}_{\text{classical}}$ classical symmetry

[Hull-Townsend]

U-duality

$G(\mathbb{R}) \circlearrowright \mathcal{M}_{\text{classical}}$ classical symmetry

Quantization of charges

[Hull-Townsend]

U-duality

$G(\mathbb{R}) \circlearrowright \mathcal{M}_{\text{classical}}$ classical symmetry

Quantization of charges $\Longrightarrow$ classical symmetry $\longrightarrow$ discrete symmetry

[Hull-Townsend]

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$G(\mathbb{R}) \circlearrowright \mathcal{M}_{\text{classical}}$ classical symmetry

$G(\mathbb{R})$ Chevalley group $G(\mathbb{Z})$

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[Hull-Townsend]

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9	$SL(2,\mathbb{R}) \times \mathbb{R}^+$	$SO(2)$	$SL(2,\mathbb{Z}) \times \mathbb{Z}_2$
8	$SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$SO(3) \times SO(2)$	$SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$
7	$SL(5,\mathbb{R})$	$SO(5)$	$SL(5,\mathbb{Z})$
6	$Spin(5,5;\mathbb{R})$	$\big(Spin(5) \times Spin(5)\big)/\mathbb{Z}_2$	$Spin(5,5;\mathbb{Z})$
5	$E_6(\mathbb{R})$	$USp(8)/\mathbb{Z}_2$	$E_6(\mathbb{Z})$
4	$E_7(\mathbb{R})$	$SU(8)/\mathbb{Z}_2$	$E_7(\mathbb{Z})$
3	$E_8(\mathbb{R})$	$Spin(16)/\mathbb{Z}_2$	$E_8(\mathbb{Z})$

[Hull-Townsend]

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Quantization of charges $\Longrightarrow$ classical symmetry $\longrightarrow$ discrete symmetry

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10	$SL(2,\mathbb{R})$	$SO(2)$	$SL(2,\mathbb{Z})$
9	$SL(2,\mathbb{R}) \times \mathbb{R}^+$	$SO(2)$	$SL(2,\mathbb{Z}) \times \mathbb{Z}_2$
8	$SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$SO(3) \times SO(2)$	$SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$
7	$SL(5,\mathbb{R})$	$SO(5)$	$SL(5,\mathbb{Z})$
6	$Spin(5,5;\mathbb{R})$	$\big(Spin(5) \times Spin(5)\big)/\mathbb{Z}_2$	$Spin(5,5;\mathbb{Z})$
5	$E_6(\mathbb{R})$	$USp(8)/\mathbb{Z}_2$	$E_6(\mathbb{Z})$
4	$E_7(\mathbb{R})$	$SU(8)/\mathbb{Z}_2$	$E_7(\mathbb{Z})$
3	$E_8(\mathbb{R})$	$Spin(16)/\mathbb{Z}_2$	$E_8(\mathbb{Z})$

All observables are invariant under $G(\mathbb{Z})$

[Hull-Townsend]

U-duality

D	$G(\mathbb{R})$	K	$G(\mathbb{Z})$
10	$SL(2,\mathbb{R})$	$SO(2)$	$SL(2,\mathbb{Z})$
9	$SL(2,\mathbb{R}) \times \mathbb{R}^+$	$SO(2)$	$SL(2,\mathbb{Z}) \times \mathbb{Z}_2$
8	$SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$SO(3) \times SO(2)$	$SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$
7	$SL(5,\mathbb{R})$	$SO(5)$	$SL(5,\mathbb{Z})$
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$$\mathcal{E}_0^{(D)}(g), \quad \mathcal{E}_4^{(D)}(g), \quad \mathcal{E}_6^{(D)}(g) \qquad : G(\mathbb{Z})\backslash G(\mathbb{R})/K \to \mathbb{C}$$

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(D) Growth: for any norm $\|\cdot\|$ on $G(\mathbb{R})$ there exists a positive integer n and constant C such that $|\varphi(g)| \leq C\|g\|^n$

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Supersymmetry constraints

bosons	SUSY	fermions
dilaton		dilatino
graviton		gravitino
…		…

[Green-Sethi]

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$$\mathcal{L}^{(3)} = \qquad \mathcal{E}_0(\tau) R^4$$

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bosons $\rightleftarrows$ SUSY $\rightleftarrows$ fermions

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$$\mathcal{L}^{(3)} = f_{12}(\tau)\lambda^{16} + f_{11}(\tau)\hat{G}\lambda^{14} + \ldots + f_0(\tau)R^4 + \ldots + f_{-12}(\tau)\lambda^{*16}$$

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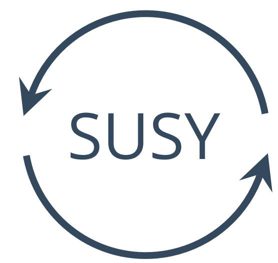
Supersymmetry constraints

bosons



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Linearized SUSY: $$f_{w+1}(\tau) = i\left(\tau_2 \frac{\partial}{\partial\tau} - i\frac{w}{2}\right) f_w(\tau)$$

[Green-Sethi]

Supersymmetry constraints

$$\int d^D x \sqrt{-g}\mathcal{L} = S = S^{(0)}$$

$$\delta\Psi = \ \delta^{(0)}\Psi$$

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Supersymmetry constraints

$$\int d^D x \sqrt{-g}\mathcal{L} = S = S^{(0)} + (\alpha')^3 S^{(3)} + (\alpha')^5 S^{(5)} + \dots$$

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Laplacian on Poincaré UHP

[Green-Sethi]

Supersymmetry constraints

bosons SUSY fermions

10 dimensions: $\Delta = 4\tau_2^2 \frac{\partial}{\partial\tau}\frac{\partial}{\partial\overline{\tau}}$ Laplacian on Poincaré UHP

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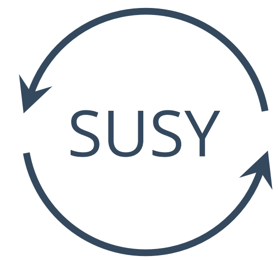
bosons  fermions

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Not an automorphic form in a strict sense

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Similarly for lower dimensions

Eisenstein series

$E(s;\tau) =$ $\qquad s \in \mathbb{C}$

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$$E(s;\tau) = \sum_{\substack{c,d\in\mathbb{Z} \\ (c,d)\neq(0,0)}} \frac{\mathrm{Im}(\tau)^s}{|c\tau + d|^{2s}} \qquad s \in \mathbb{C}$$

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$$B(\mathbb{Z}) = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in SL(2,\mathbb{Z}) \right\}$$

Borel subgroup

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

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$$\chi : B(\mathbb{Z})\backslash B(\mathbb{R}) \to \mathbb{C}^\times$$

Multiplicative character trivially extended to $G(\mathbb{R})$

$$\tau \mapsto \mathrm{Im}(\tau)^s$$

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$$E(s;\tau) = \tau_2^s + \frac{\xi(2s-1)}{\xi(2s)}\tau_2^{1-s} + \frac{2\tau_2^{1/2}}{\xi(2s)} \sum_{m\neq 0} |m|^{s-1/2}\,\sigma_{1-2s}(m) K_{s-1/2}(2\pi\,|m|\,\tau_2) e^{2\pi i m \tau_1}$$

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$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$$

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Divisor sum

$$\sigma_s(m) = \sum_{d\,|\,m} d^s$$

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Completed Riemann zeta function

$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$

Divisor sum

$\sigma_s(m) = \sum_{d\,|\,m} d^s$

Bessel function of the second kind

Eisenstein series

$$E(s;\tau) = \sum_{\gamma \in B(\mathbb{Z})\backslash SL(2,\mathbb{Z})} \chi(\gamma(\tau))$$

$$\big(\Delta - s(s-1)\big)E(s;\tau) = 0$$

$$E(s;\gamma(\tau)) = E(s;\tau) \qquad E(s;\tau+1) = E(s;\tau)$$

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$$\big(\Delta - s(s-1)\big)E(s;\tau) = 0 \qquad E(s;\tau) \sim \tau_2^s \qquad g_{\rm s} = \tau_2^{-1} \to 0$$

[Green-Gutperle, Pioline, Green-Russo-Vanhove]

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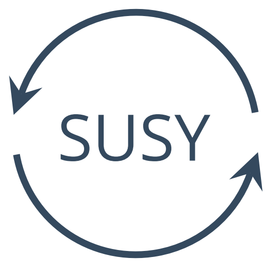

$$(\Delta - \tfrac{3}{4})\mathcal{E}_0(\tau) = 0$$

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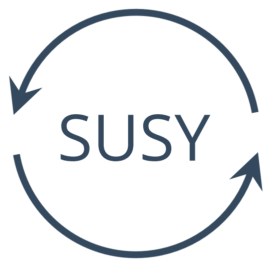

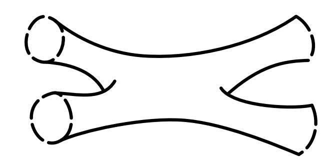

(Einstein frame)

$$(\Delta - \tfrac{3}{4})\mathcal{E}_0(\tau) = 0 \qquad \mathcal{E}_0(\tau) \sim 2\zeta(3)\tau_2^{3/2}$$

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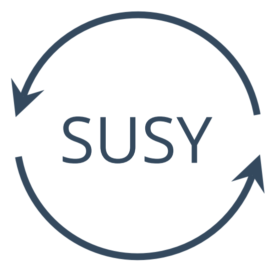

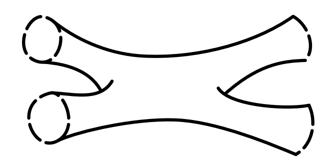

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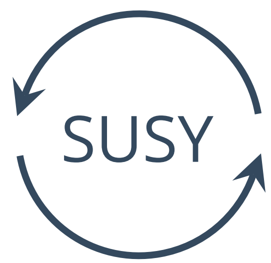

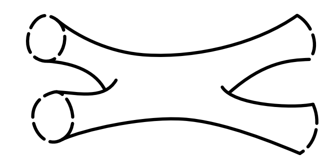

$$\mathcal{E}_0(\tau) = 2\zeta(3)E(3/2;\tau)$$

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$\mathcal{E}_6(\tau)$ as a sum over images $\sum_{B(\mathbb{Q})\backslash G(\mathbb{Z})}$ but not of a character χ

[Green-Miller-Vanhove]

Extracting physical information

Expand Bessel function in g_{s} $\tau = \chi + i g_{\mathrm{s}}^{-1}$

[Green-Gutperle]

Extracting physical information

Expand Bessel function in g_s $\qquad\qquad$ $\tau = \chi + i g_\mathrm{s}^{-1}$

$$\mathcal{E}_0(\tau) = 2\zeta(3) g_\mathrm{s}^{-3/2} + 4\zeta(2) g_\mathrm{s}^{1/2} + 2\pi \sum_{m\neq 0} \sqrt{|m|}\sigma_{-2}(m) e^{-2\pi|m|g_\mathrm{s}^{-1} + 2\pi i m \chi}\left[1 + \mathcal{O}(g_\mathrm{s})\right]$$

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Expand Bessel function in $g_{\rm s}$ $\qquad$ $\tau = \chi + i g_{\rm s}^{-1}$

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Perturbative
(zero-mode)

[Green-Gutperle]

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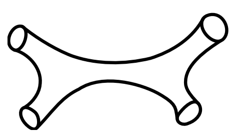 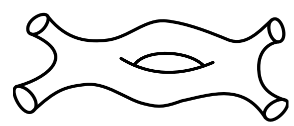

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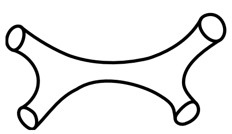 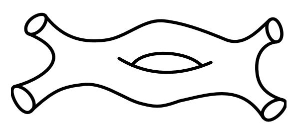

$$\mathcal{E}_0(\tau) = \underbrace{2\zeta(3) g_{\mathrm{s}}^{-3/2} + 4\zeta(2) g_{\mathrm{s}}^{1/2}}_{\text{Perturbative (zero-mode)}} + \underbrace{2\pi \sum_{m\neq 0} \sqrt{|m|}\sigma_{-2}(m) e^{-2\pi|m|g_{\mathrm{s}}^{-1} + 2\pi i m\chi}\left[1 + \mathcal{O}(g_{\mathrm{s}})\right]}_{\text{Non-perturbative (remaining modes)}}$$

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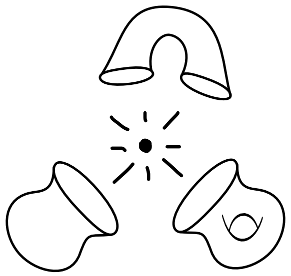

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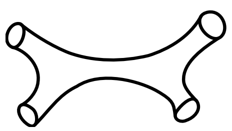

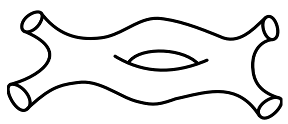

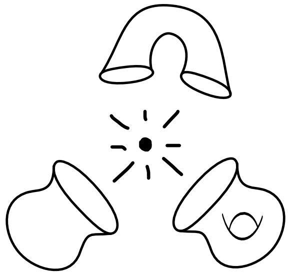

Instanton action

$$\mathcal{E}_0(\tau) = \underbrace{2\zeta(3) g_{\mathrm{s}}^{-3/2} + 4\zeta(2) g_{\mathrm{s}}^{1/2}}_{\text{Perturbative (zero-mode)}} + \underbrace{2\pi \sum_{m\neq 0} \sqrt{|m|}\,\sigma_{-2}(m)\, e^{-2\pi|m|g_{\mathrm{s}}^{-1} + 2\pi i m\chi}\left[1 + \mathcal{O}(g_{\mathrm{s}})\right]}_{\text{Non-perturbative (remaining modes)}}$$

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Non-perturbative
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Sums over the number of ways the charge *m* can be factorised into two integers

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Sums over the number of ways the charge *m* can be factorised into two integers

wrapping number of a T-dual D-particle and charge

[Green-Gutperle]

Lower dimensions

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D	$G(\mathbb{R})$	K	$G(\mathbb{Z})$
10	$SL(2,\mathbb{R})$	$SO(2)$	$SL(2,\mathbb{Z})$
9	$SL(2,\mathbb{R}) \times \mathbb{R}^+$	$SO(2)$	$SL(2,\mathbb{Z}) \times \mathbb{Z}_2$
8	$SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$SO(3) \times SO(2)$	$SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$
7	$SL(5,\mathbb{R})$	$SO(5)$	$SL(5,\mathbb{Z})$
6	$Spin(5,5;\mathbb{R})$	$\big(Spin(5) \times Spin(5)\big)/\mathbb{Z}_2$	$Spin(5,5;\mathbb{Z})$
5	$E_6(\mathbb{R})$	$USp(8)/\mathbb{Z}_2$	$E_6(\mathbb{Z})$
4	$E_7(\mathbb{R})$	$SU(8)/\mathbb{Z}_2$	$E_7(\mathbb{Z})$
3	$E_8(\mathbb{R})$	$Spin(16)/\mathbb{Z}_2$	$E_8(\mathbb{Z})$

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$$E(\chi; g) = \sum_{\gamma \in B(\mathbb{Z})\backslash G(\mathbb{Z})} \chi(\gamma g)$$

Parabolic subgroups

Fourier expand
in different directions $\longleftrightarrow$ Unipotent subgroup U

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Choice of parabolic subgroup P

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Σ choice of simple roots $\quad$ $\langle \Sigma \rangle$ generated root system

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Choice of parabolic subgroup P

Σ choice of simple roots $\langle\Sigma\rangle$ generated root system

$G = SL(4)$  $\Sigma = \{\alpha_1\}$

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Choice of parabolic subgroup P

Σ choice of simple roots

$\langle \Sigma \rangle$ generated root system

$G = SL(4)$

$\Sigma = \{\alpha_1\}$

$$L = \left\{ \begin{pmatrix} * & * & & \\ * & * & & \\ & & * & \\ & & & * \end{pmatrix} \right\} \quad U = \left\{ \begin{pmatrix} 1 & & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} \right\}$$

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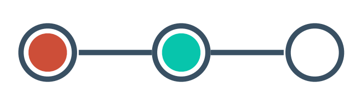
$G = SL(4)$  $\Sigma = \{\alpha_1\}$

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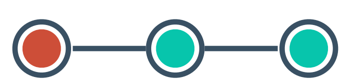
$G = SL(4)$  $\Sigma = \{\alpha_1\}$

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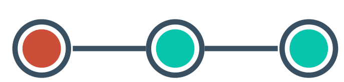
$G = SL(4)$  $\Sigma = \{\alpha_1\}$

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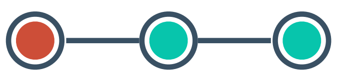

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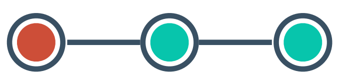

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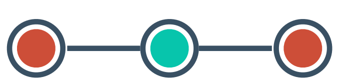

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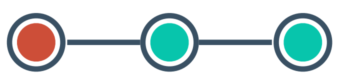

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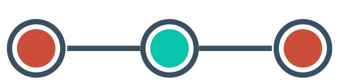

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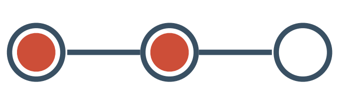

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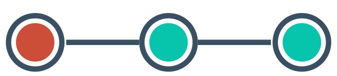

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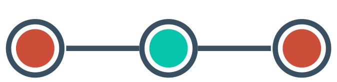

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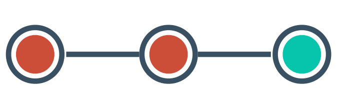

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Parabolic subgroups

Parabolic subgroups

Maximal parabolic

Parabolic subgroups

Minimal parabolic
Borel

Maximal parabolic

Parabolic subgroups

Minimal parabolic
Borel

$$B = NA$$

$$N = \left\{ \begin{pmatrix} \boxed{1} & * & * & * \\ & \boxed{1} & * & * \\ & & \boxed{1} & * \\ & & & \boxed{1} \end{pmatrix} \right\}$$

Maximal parabolic

$$P = LU$$

$$U = \left\{ \begin{pmatrix} 1 & & & * \\ & 1 & & * \\ & & 1 & * \\ & & & \boxed{1} \end{pmatrix} \right\}$$

Fourier expansion

Fourier expansion

Let $\psi : U(\mathbb{Z})\backslash U(\mathbb{R}) \to \overset{\cong S^1}{U(1)}$ be a multiplicative character

Parametrised by $m_\alpha \in \mathbb{Z}$ called charges

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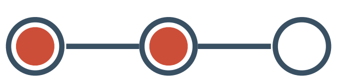

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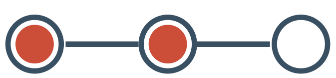

$$\psi_U\left(\begin{pmatrix} 1 & & & y_1 \\ & 1 & & y_2 \\ & & 1 & y_3 \\ & & & 1 \end{pmatrix}\right) = e^{2\pi i(m_1 y_1 + m_2 y_2 + m_3 y_3)}$$

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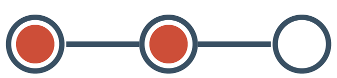

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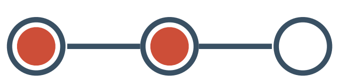

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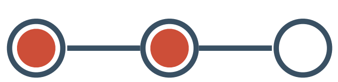
Fourier expansion

$\psi(u_1u_2) = \psi(u_1)\psi(u_2)$

Let $\psi : U(\mathbb{Z})\backslash U(\mathbb{R}) \to \overset{\overset{S^1}{\cong}}{U(1)}$ be a multiplicative character

Parametrised by $m_\alpha \in \mathbb{Z}$ called charges



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Fourier expansion

$\cong S^1$ $\qquad$ $\psi(u_1u_2) = \psi(u_1)\psi(u_2)$

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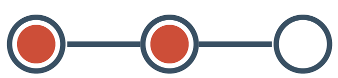

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$$F_U(\chi, \psi; g) = \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} E(\chi, ug)\overline{\psi(u)}\, du$$

Fourier expansion

Fourier expansion

$$E(\chi; g) = \sum_{\psi} F_U(\chi, \psi; g)$$

Fourier expansion

$$E(\chi; g) = F_U(\chi, 1; g) + \sum_{\psi \neq 1} F_U(\chi, \psi; g)$$

Fourier expansion

$$E(\chi; g) = F_U(\chi, 1; g) + \sum_{\psi \neq 1} F_U(\chi, \psi; g)$$

$$F_U(\chi, \psi; ug) = \psi(u) F_U(\chi, \psi; g) \qquad \psi(u_1 u_2) = \psi(u_1)\psi(u_2)$$

Fourier expansion

$$E(\chi;g) = F_U(\chi,1;g) + \sum_{\psi^{(1)}\neq 1} F_{U^{(1)}}(\chi,\psi^{(1)};g) + \sum_{\psi^{(2)}\neq 1} F_{U^{(2)}}(\chi,\psi^{(2)};g) + \dots$$

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$$U^{(1)} = U \qquad U^{(n+1)} = [U^{(n)}, U^{(n)}]$$

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$N :$ $\qquad \psi^{(1)}\left(\begin{pmatrix} 1 & x_1 & * & * \\ & 1 & x_2 & * \\ & & 1 & x_3 \\ & & & 1 \end{pmatrix}\right)$

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$$N: \qquad \psi^{(1)}\left(\begin{pmatrix} 1 & x_1 & * & * \\ & 1 & x_2 & * \\ & & 1 & x_3 \\ & & & 1 \end{pmatrix}\right) \qquad \psi^{(2)}\left(\begin{pmatrix} 1 & 0 & z_1 & z_2 \\ & 1 & 0 & z_3 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}\right)$$

$$F_U(\chi,\psi;ug) = \psi(u)F_U(\chi,\psi;g) \qquad \psi(u_1u_2) = \psi(u_1)\psi(u_2)$$

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Terminology

$P = B \longrightarrow U = N$ Fourier coefficient is a Whittaker coefficient

$$N = \left\{ \begin{pmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} \right\}$$

Terminology

$P = B \longrightarrow U = N$ Fourier coefficient is a Whittaker coefficient

$N = \left\{ \begin{pmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} \right\}$ F_U W_N

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$P = B \longrightarrow U = N$ Fourier coefficient is a Whittaker coefficient

$N = \left\{ \begin{pmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} \right\}$ $\qquad F_U \qquad W_N$

Characters and coefficients with all $m_\alpha \neq 0$ are called generic otherwise they are called degenerate

Fourier expansion

Choice of unipotent subgroup U $\longleftrightarrow$ Study different perturbative and non-perturbative effects

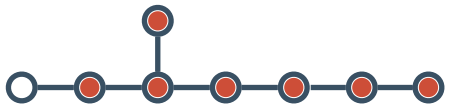
[Green-Miller-Vanhove]

Fourier expansion

Choice of unipotent subgroup U $\longleftrightarrow$ Study different perturbative and non-perturbative effects

- String perturbation limit $g_{\mathrm{s}} \to 0$

  D-instantons | NS5-instantons



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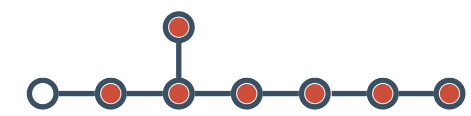
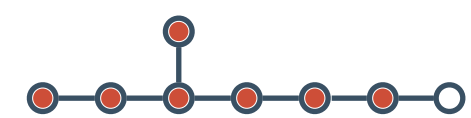
  

- Decompactification limit
  Higher dimensional black holes | BPS states

  Large radius for compactified circle

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  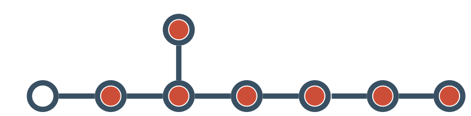

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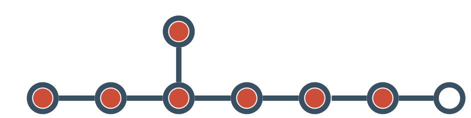
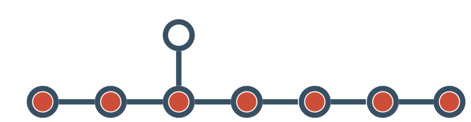
  

- **M-theory limit**
  M2, M5-instantons

  Large M-theory torus

[Green-Miller-Vanhove]

Fourier expansion

Choice of unipotent subgroup U $\longleftrightarrow$ Study different perturbative and non-perturbative effects

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  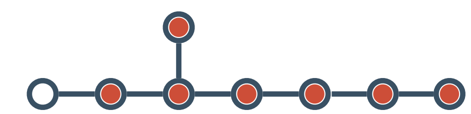
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  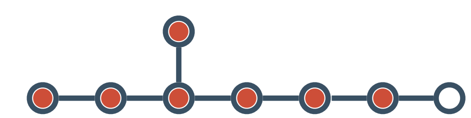
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  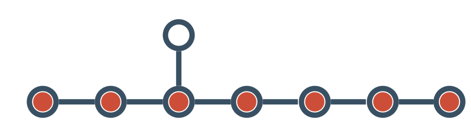

[Green-Miller-Vanhove]

Maximal parabolic subgroups

Fourier expansion

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  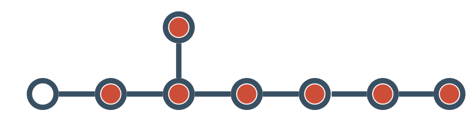

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  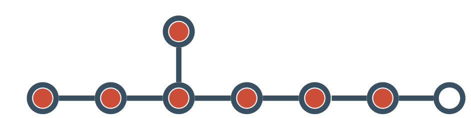

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  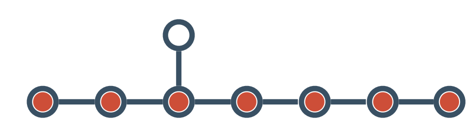

[Green-Miller-Vanhove]

Maximal parabolic subgroups

Difficult to compute!

Fourier expansion

Goal: find expressions for Fourier coefficients in terms of (known) Whittaker coefficients

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Fourier expansion

Goal: find expressions for Fourier coefficients
in terms of (known) Whittaker coefficients

Would allow us to compute non-perturbative effects that capture information about instantons and black holes

Adelic framework

> *An efficient, but abstract, way to approach the subject of automorphic forms is by the introduction of adeles, rather ungainly objects that nevertheless, once familiar, spare much unnecessary thought and many useless calculations.*

— Robert P. Langlands*

*Representation theory - its rise and its role in number theory, Proceedings of the Gibbs symposium (1989)

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Eisenstein series

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Adelic Eisenstein series

$\uparrow$ Lift

Eisenstein series

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$$\begin{tikzcd} \text{Adelic Eisenstein series} \arrow[r, "\text{Compute}"] & \text{Adelic Fourier coefficient} \\ \text{Eisenstein series} \arrow[u, "\text{Lift}"'] & \end{tikzcd}$$

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$$\begin{tikzcd}
\text{Adelic Eisenstein series} \arrow[r, "\text{Compute}"] & \text{Adelic Fourier coefficient} \arrow[d, "\text{Restrict}"] \\
\text{Eisenstein series} \arrow[u, "\text{Lift}"'] & \text{Fourier coefficient}
\end{tikzcd}$$

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The adeles

For a prime p

$\mathbb{Q}$

The adeles

For a prime p

$$\begin{tikzcd} \mathbb{Q} \arrow[r, bend left, "\text{Euclidean norm}"] & \mathbb{R} \end{tikzcd}$$

The adeles

For a prime p

$$\begin{tikzcd}
 & \mathbb{R} \\
\mathbb{Q} \arrow[ur, "\text{Euclidean norm}"] \arrow[dr, "p\text{-adic norm}"'] & \\
 & \mathbb{Q}_p
\end{tikzcd}$$

p-adic numbers

The adeles

For a prime p

$$
\begin{tikzcd}
& \mathbb{R} \\
\mathbb{Q} \arrow[ur, "\text{Euclidean norm}"] \arrow[r, "\text{p-adic norm}"'] & \mathbb{Q}_p
\end{tikzcd}
$$

p-adic numbers

$$|p_1^{k_1} \cdots p_r^{k_r}|_{p_i} = p_i^{-k_i}$$

The adeles

For a prime p

$$\mathbb{Q} \xrightarrow{\text{Euclidean norm}} \mathbb{R} \qquad \mathbb{R} = \mathbb{Q}_\infty$$

$$\mathbb{Q} \xrightarrow[p\text{-adic norm}]{} \mathbb{Q}_p \qquad p\text{-adic numbers}$$

$$|p_1^{k_1} \cdots p_r^{k_r}|_{p_i} = p_i^{-k_i}$$

The adeles

For a prime *p*

$$\mathbb{Q} \begin{cases} \xrightarrow{\text{Euclidean norm}} \mathbb{R} & \mathbb{R} = \mathbb{Q}_\infty \\ \xrightarrow[\text{p-adic norm}]{} \mathbb{Q}_p & \text{p-adic numbers} \end{cases}$$

$$|p_1^{k_1} \cdots p_r^{k_r}|_{p_i} = p_i^{-k_i}$$

The adeles are then defined as

$$\mathbb{A} = \mathbb{A}_{\mathbb{Q}} = \mathbb{R} \times \prod_{p \text{ prime}}{}' \mathbb{Q}_p \qquad x = (x_\infty; x_2, x_3, x_5, \ldots) \in \mathbb{A}$$

The adeles

For a prime *p*

$$\begin{tikzcd} & \mathbb{R} \\ \mathbb{Q} \arrow[ur, "\text{Euclidean norm}"] \arrow[dr, "p\text{-adic norm}"'] & \\ & \mathbb{Q}_p \end{tikzcd}$$

$\mathbb{Q}_p$ *p*-adic numbers

$\mathbb{R} = \mathbb{Q}_\infty$

$|p_1^{k_1} \cdots p_r^{k_r}|_{p_i} = p_i^{-k_i}$

The adeles are then defined as

$$\mathbb{A} = \mathbb{A}_\mathbb{Q} = \mathbb{R} \times \prod_{p \text{ prime}}{}' \mathbb{Q}_p \qquad x = (x_\infty; x_2, x_3, x_5, \ldots) \in \mathbb{A}$$

$$\mathbb{Q} \hookrightarrow \mathbb{A}$$
$$q \mapsto (q; q, q, \ldots)$$

$\mathbb{Q}$ is discrete in $\mathbb{A}$ taking the role of $\mathbb{Z}$ in $\mathbb{R}$

Much easier to work with since it is a field!

Adelic framework

$$\mathcal{E}_0^{(D)}(g), \quad \mathcal{E}_4^{(D)}(g), \quad \mathcal{E}_6^{(D)}(g) \qquad : G(\mathbb{Z})\backslash G(\mathbb{R})/K \to \mathbb{C}$$

Adelic framework

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Lift to the adeles [FGKP15 §4.2.2]

$$G(\mathbb{A}) = G(\mathbb{R}) \times \prod_{p \text{ prime}}{}' G(\mathbb{Q}_p) \qquad K_{\mathbb{A}} = K \times \prod_{p \text{ prime}} G(\mathbb{Z}_p)$$

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Adelic framework

Adelic framework

Eisenstein series $\longrightarrow$ Adelic Eisenstein series

Adelic framework

Eisenstein series $\longrightarrow$ Adelic Eisenstein series

$$\sum_{\gamma \in B(\mathbb{Z})\backslash G(\mathbb{Z})} \chi_{\mathbb{R}}(\gamma g) \qquad \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} \chi_{\mathbb{A}}(\gamma g)$$

Adelic framework

Eisenstein series $\longrightarrow$ Adelic Eisenstein series

$$\sum_{\gamma \in B(\mathbb{Z})\backslash G(\mathbb{Z})} \chi_{\mathbb{R}}(\gamma g) \qquad\qquad \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} \chi_{\mathbb{A}}(\gamma g)$$

Fourier coefficients $\longrightarrow$ Adelic Fourier coefficients

Adelic framework

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Fourier coefficients $\longrightarrow$ Adelic Fourier coefficients

$$\int_{U(\mathbb{Z})\backslash U(\mathbb{R})} E(\chi; ug)\overline{\psi_{\mathbb{R}}(u)}\, du \qquad\qquad \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} E(\chi; ug)\overline{\psi_{\mathbb{A}}(u)}\, du$$

$$m_\alpha \in \mathbb{Z} \qquad\qquad m_\alpha \in \mathbb{Q}$$

Computing adelic Fourier coefficients

[FGKP15 §9-10]

Whittaker coefficients

Computing adelic Fourier coefficients

[FGKP15 §9-10]

Whittaker coefficients

Constant term: Langlands' constant term formula

Computing adelic Fourier coefficients

[FGKP15 §9-10]

Whittaker coefficients

Constant term: Langlands' constant term formula

Generic coefficient: Factorises over the primes. Casselman-Shalika formula

Computing adelic Fourier coefficients

[FGKP15 §9-10]

Whittaker coefficients

Constant term: Langlands' constant term formula

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Degenerate coefficient: Reduction to generic coefficient on smaller group

Computing adelic Fourier coefficients

[FGKP15 §9-10]

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[FGKP15 §9-10]

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Factorisation

Computing adelic Fourier coefficients

[FGKP15 §9-10]

Whittaker coefficients

Constant term: Langlands' constant term formula

Generic coefficient: Factorises over the primes. Casselman-Shalika formula

Degenerate coefficient: Reduction to generic coefficient on smaller group

Maximally degenerate   Factorisation

[GKP14]

Fourier coefficients

In terms of Whittaker coefficients
Simplify drastically for certain χ

Example of simplifications

$G = SL(3)$ $\qquad E(\chi; g)$ $\qquad \chi \longleftrightarrow (s_1, s_2) \in \mathbb{C}^2$

[FGKP15 §10.6]

Example of simplifications

$$G = SL(3) \qquad E(\chi; g) \qquad \chi \longleftrightarrow (s_1, s_2) \in \mathbb{C}^2$$

$$N = \left\{ \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \right\} \qquad \psi_{m_1,m_2}\left(\begin{pmatrix} 1 & x_1 & * \\ & 1 & x_2 \\ & & 1 \end{pmatrix}\right) = e^{2\pi i(m_1 x_1 + m_2 x_2)}$$

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$$W_N(\chi, \psi_{m_1,m_2}; g) \propto \left(\begin{smallmatrix}\text{arithmetic} \\ \text{factor}\end{smallmatrix}\right) \int K_{\#}(\ldots)\, K_{\#}(\ldots)$$

[FGKP15 §10.6]

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p-adic part

$$W_N(\chi, \psi_{m_1,m_2}; g) \propto \underbrace{\left(\begin{smallmatrix}\text{arithmetic}\\ \text{factor}\end{smallmatrix}\right)}_{} \int K_\#(\ldots)\, K_\#(\ldots)$$

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$$W_N(\chi, \psi_{m_1,m_2}; g) \propto \binom{\text{arithmetic}}{\text{factor}} \int K_{\#}(\dots)\, K_{\#}(\dots)$$

Vanishes for certain (s_1, s_2)

[FGKP15 §10.6]

Example of simplifications

$$W_N(\chi, \psi_{m_1,m_2}; g) \propto \left(\substack{\text{arithmetic}\\\text{factor}}\right) \int K_\#(\ldots)\, K_\#(\ldots)$$

[FGKP15 §10.6]

Example of simplifications

Certain (s_1, s_2)

$$W_N(\chi, \psi_{m_1,m_2}; g) \propto \left(\substack{\text{arithmetic}\\ \text{factor}}\right) \int K_\#(\ldots)\, K_\#(\ldots)$$

[FGKP15 §10.6]

Example of simplifications

Certain (s_1, s_2)

$$W_N(\chi, \psi_{m_1,m_2}; g) \propto \left(\substack{\text{arithmetic}\\\text{factor}}\right) \int K_\#(\ldots)\, K_\#(\ldots) \longrightarrow 0$$

[FGKP15 §10.6]

Example of simplifications

Certain (s_1, s_2)

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$$W_N(\chi, \psi_{m_1,0}; g) \propto K_\#(\ldots) + K_\#(\ldots) + K_\#(\ldots)$$

[FGKP15 §10.6]

Example of simplifications

Certain (s_1, s_2)

$$W_N(\chi, \psi_{m_1,m_2}; g) \propto \left(\substack{\text{arithmetic}\\\text{factor}}\right) \int K_\#(\ldots)\, K_\#(\ldots) \quad \longrightarrow \quad 0$$

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Example of simplifications

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$$W_N(\chi, \psi_{m_1,0}; g) \propto K_\#(\ldots) + K_\#(\ldots) + K_\#(\ldots) \longrightarrow K_\#(\ldots)$$

To explain this, we need to study
small automorphic representations

[FGKP15 §10.6]

Automorphic representations

$G(\mathbb{A}) \circlearrowright$ Space of automorphic forms*

* With some subtleties described in [FGKP15 §6]

[Bump, Goldfeld-Hundley]

Automorphic representations

$G(\mathbb{A}) \circlearrowright$ Space of automorphic forms*

Automorphic representation π = an irreducible component of the above space under this action

* With some subtleties described in [FGKP15 §6]

[Bump, Goldfeld-Hundley]

Automorphic representations

$G(\mathbb{A}) \circlearrowright$ Space of automorphic forms*

Automorphic representation π = an irreducible component of the above space under this action

What is a small automorphic representation?

* With some subtleties described in [FGKP15 §6]

[Bump, Goldfeld-Hundley]

Wavefront set

[Mœglin–Waldspurger, Matumoto, Ginzburg-Rallis-Soudry, Ginzburg, Gomez-Gourevitch-Sahi, Jiang-Liu-Savin, Joseph, Miller-Sahi]

Wavefront set

The (global) wavefront set contains all the characters ψ which can give rise to non-vanishing Fourier coefficients in that representation

[Mœglin–Waldspurger, Matumoto, Ginzburg-Rallis-Soudry, Ginzburg, Gomez-Gourevitch-Sahi, Jiang-Liu-Savin, Joseph, Miller-Sahi]

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$$\psi \notin \mathrm{WF}(\pi) \implies F_U(\chi, \psi; g) = 0 \quad \text{for} \quad E(\chi; g) \in \pi$$

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Small automorphic representations have few non-vanishing Fourier coefficients

[Mœglin–Waldspurger, Matumoto, Ginzburg-Rallis-Soudry, Ginzburg, Gomez-Gourevitch-Sahi, Jiang-Liu-Savin, Joseph, Miller-Sahi]

Wavefront set

Characters ψ $\longleftrightarrow$ Nilpotent elements in $\mathfrak{g}(\mathbb{Q})$

[Mœglin–Waldspurger, Matumoto, Ginzburg-Rallis-Soudry, Ginzburg, Gomez-Gourevitch-Sahi, Jiang-Liu-Savin, Joseph, Miller-Sahi]

Wavefront set

Characters ψ $\longleftrightarrow$ Nilpotent elements in $\mathfrak{g}(\mathbb{Q})$

Nilpotent orbit $\mathcal{O} = \{gXg^{-1} \mid g \in G(\mathbb{C})\}$ $X \in \mathfrak{g}$ nilpotent

[Mœglin–Waldspurger, Matumoto, Ginzburg-Rallis-Soudry, Ginzburg, Gomez-Gourevitch-Sahi, Jiang-Liu-Savin, Joseph, Miller-Sahi]

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$$\mathrm{WF}(\pi) = \bigcup_i \overline{\mathcal{O}_i}$$

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So called special orbits

[Mœglin–Waldspurger, Matumoto, Ginzburg-Rallis-Soudry, Ginzburg, Gomez-Gourevitch-Sahi, Jiang-Liu-Savin, Joseph, Miller-Sahi]

Wavefront set

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Closure with respect to partial ordering

$$\mathrm{WF}(\pi) = \bigcup_i \overline{\mathcal{O}_i}$$

So called special orbits

[Mœglin–Waldspurger, Matumoto, Ginzburg-Rallis-Soudry, Ginzburg, Gomez-Gourevitch-Sahi, Jiang-Liu-Savin, Joseph, Miller-Sahi]

Nilpotent orbits

[Collingwood-McGovern]

For $SL(n)$, orbits can be identified with partitions of n

Nilpotent orbits

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For $SL(n)$, orbits can be identified with partitions of n

$$(p_1, p_2, \ldots)$$

Nilpotent orbits

[Collingwood-McGovern]

For $SL(n)$, orbits can be identified with partitions of n

decreasing order

$(p_1, p_2, \dots)$

Nilpotent orbits

[Collingwood-McGovern]

For $SL(n)$, orbits can be identified with partitions of n

decreasing order

$$(p_1, p_2, \ldots) \leq (q_1, q_2, \ldots)$$

partial ordering

Nilpotent orbits

[Collingwood-McGovern]

For $SL(n)$, orbits can be identified with partitions of n

decreasing order

$$(p_1, p_2, \ldots) \leq (q_1, q_2, \ldots)$$ partial ordering

$$\Longleftrightarrow$$

$$\sum_{i=1}^{k} p_i \leq \sum_{i=1}^{k} q_i \quad \forall k$$

Nilpotent orbits

[Collingwood-McGovern]

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Illustrated by a Hasse diagram

Nilpotent orbits

[Collingwood-McGovern]

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partial ordering

$$\Longleftrightarrow$$

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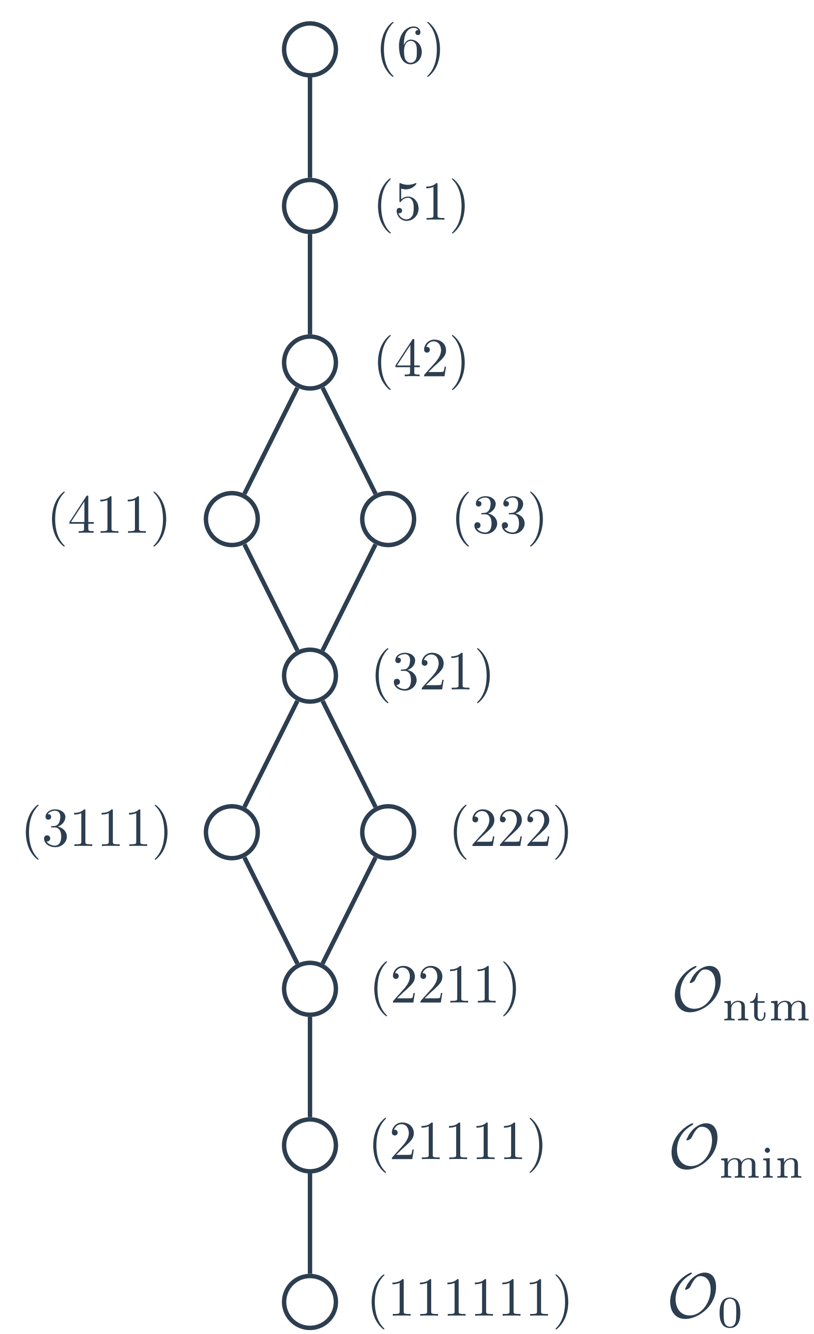
Illustrated by a Hasse diagram

$SL(6)$



Nilpotent orbits

[Collingwood-McGovern]

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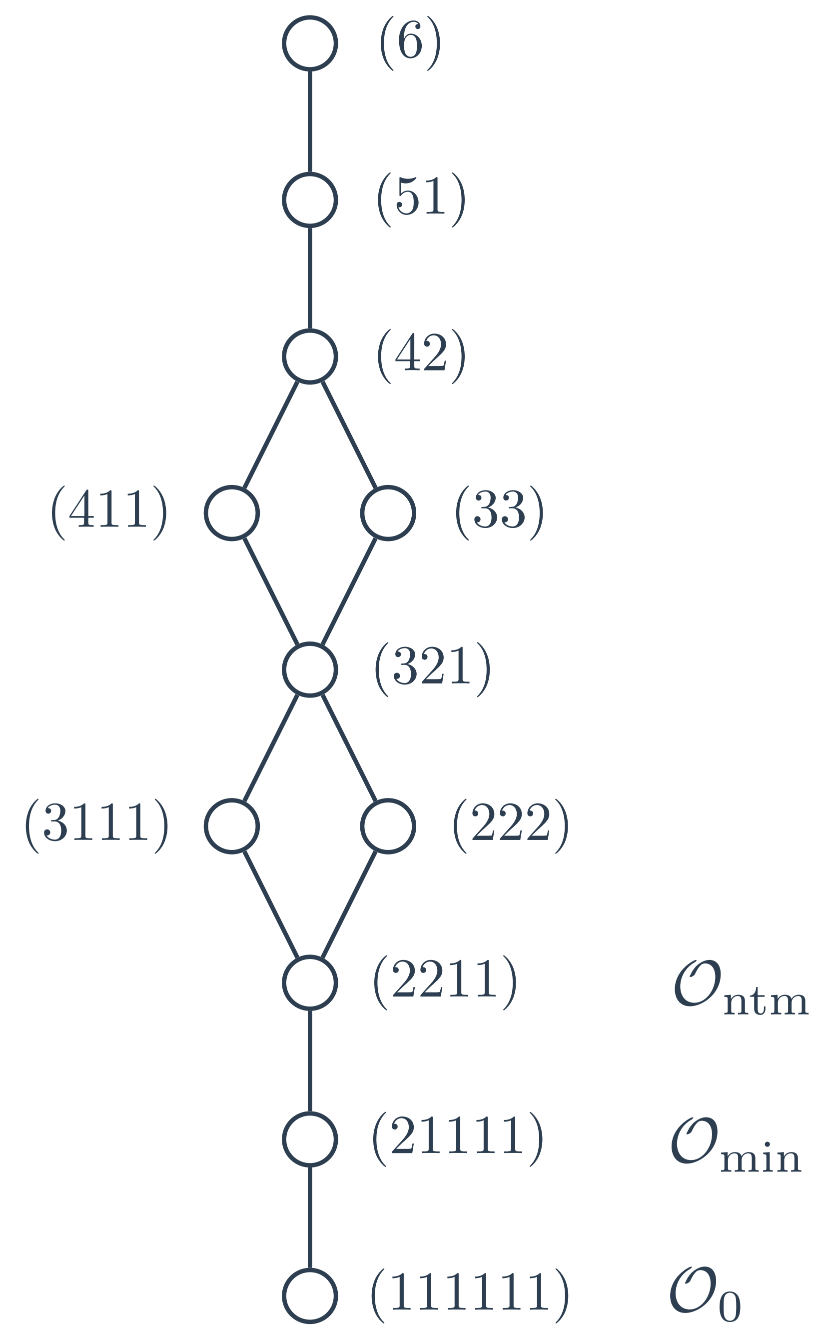
Illustrated by a Hasse diagram

Closure: $\overline{\mathcal{O}} = \bigcup_{\mathcal{O}' \le \mathcal{O}} \mathcal{O}'$

$SL(6)$



Automorphic representations

Small representations

Automorphic representations

Small representations

$$\mathrm{WF}(\pi_{\mathrm{min}}) = \overline{\mathcal{O}_{\mathrm{min}}} = \mathcal{O}_{\mathrm{min}} \cup \mathcal{O}_0$$

$$\mathrm{WF}(\pi_{\mathrm{ntm}}) = \overline{\mathcal{O}_{\mathrm{ntm}}} = \mathcal{O}_{\mathrm{ntm}} \cup \mathcal{O}_{\mathrm{min}} \cup \mathcal{O}_0$$

Automorphic representations

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$\mathcal{E}_0^{(D)} \in \pi_{\mathrm{min}}$ $\mathcal{E}_4^{(D)} \in \pi_{\mathrm{ntm}}$ [Green-Miller-Vanhove, Pioline, Bossard-Verschinin]

Automorphic representations

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χ_{min} such that $E(\chi_{\mathrm{min}}, g) \in \pi_{\mathrm{min}}$

Automorphic representations

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$\mathcal{E}_0^{(D)} \in \pi_{\min}$ $\mathcal{E}_4^{(D)} \in \pi_{\mathrm{ntm}}$ [Green-Miller-Vanhove, Pioline, Bossard-Verschinin]

Certain (s_1, s_2) $\longleftrightarrow$ $\chi_{\min}$ such that $E(\chi_{\min}, g) \in \pi_{\min}$

Automorphic representations

Small representations

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$\int K\,K \longrightarrow 0$

$\sum K \longrightarrow K$

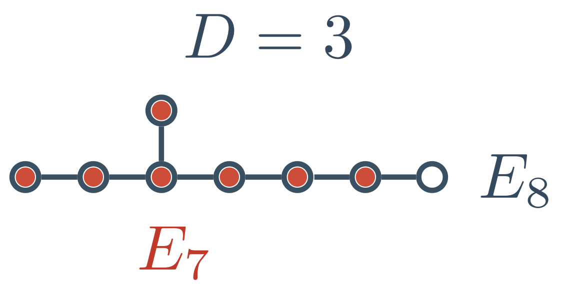
Automorphic representations

[Ferrara-Günaydin, Ferrara-Maldacena, Green-Miller-Vanhove]

Automorphic representations

- Decompactification limit
  Higher dimensional black holes | BPS states

Large radius for
compactified circle



[Ferrara-Günaydin, Ferrara-Maldacena, Green-Miller-Vanhove]

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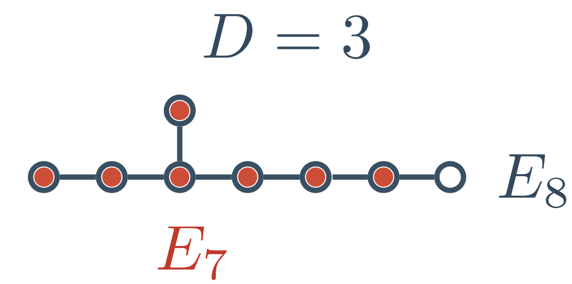

π_{min} π_{ntm} π_{3A_1} π_{A_2}

[Ferrara-Günaydin, Ferrara-Maldacena, Green-Miller-Vanhove]

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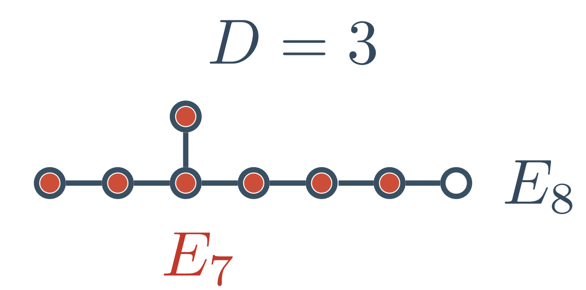

$\mathcal{E}_0^{(D)}$ $\mathcal{E}_4^{(D)}$

π_{min} π_{ntm} π_{3A_1} π_{A_2}

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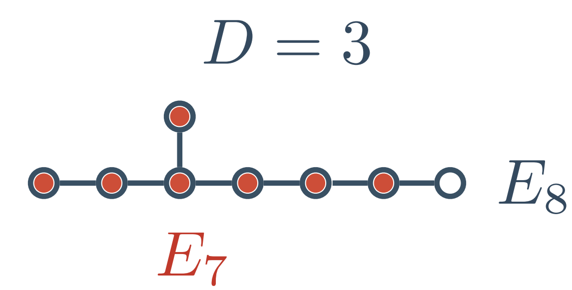

	$\mathcal{E}_0^{(D)}$	$\mathcal{E}_4^{(D)}$			
	π_{min}	π_{ntm}	π_{3A_1}	π_{A_2}	
$\dim\{\psi_U \in \mathrm{WF}\}$	28	45	55	56	[Miller-Sahi]

[Ferrara-Günaydin, Ferrara-Maldacena, Green-Miller-Vanhove]

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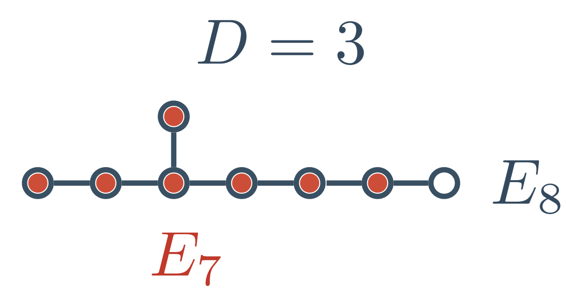

	$\mathcal{E}_0^{(D)}$ $\pi_{\min}$	$\mathcal{E}_4^{(D)}$ π_{ntm}	π_{3A_1}	π_{A_2}	
$\dim\{\psi_U \in \mathrm{WF}\}$	28	45	55	56	[Miller-Sahi]
$D=4$ BPS-orbits $E_7 \circlearrowleft \{\psi_U\}$	½ BPS	¼ BPS	⅛ BPS	⅛ BPS$^+$	

[Ferrara-Günaydin, Ferrara-Maldacena, Green-Miller-Vanhove]

Goal: find expressions for Fourier coefficients
in terms of (known) Whittaker coefficients
using vanishing properties of the given π

Previous results

[Miller-Sahi]

Previous results

Theorem

For $G = E_6, E_7$, an automorphic form $\varphi \in \pi_{\min}$ is completely determined by maximally degenerate Whittaker coefficients

W_N with only one $m_\alpha \neq 0$

[Miller-Sahi]

Main results

$SL(3), SL(4)$

[GKP14]

Main results $SL(3), SL(4)$

Theorem

For $G = SL(3), SL(4)$, an automorphic form $\varphi \in \pi_{\min}$ is completely determined by maximally degenerate Whittaker coefficients.

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More generally, for $\varphi \in \pi$

[GKP14]

Main results $SL(3), SL(4)$

Theorem

For $G = SL(3), SL(4)$, an automorphic form $\varphi \in \pi_{\mathrm{min}}$ is completely determined by maximally degenerate Whittaker coefficients.

More generally, for $\varphi \in \pi$

$$\varphi = \sum_{\mathcal{O}} \varphi_{\mathcal{O}} \qquad \text{where } \varphi_{\mathcal{O}} \text{ vanishes unless } \mathcal{O} \subseteq \mathrm{WF}(\pi)$$

[GKP14]

Main results $SL(3), SL(4)$

$$\varphi = \sum_{\mathcal{O}} \varphi_{\mathcal{O}}$$ where $\varphi_{\mathcal{O}}$ vanishes unless $\mathcal{O} \subseteq \mathrm{WF}(\pi)$

Corollary

[GKP14]

Main results $SL(3), SL(4)$

$$\varphi = \sum_{\mathcal{O}} \varphi_{\mathcal{O}}$$ where $\varphi_{\mathcal{O}}$ vanishes unless $\mathcal{O} \subseteq \mathrm{WF}(\pi)$

Corollary

$\varphi \in \pi_{\mathrm{min}}$ maximally degenerate Whittaker coefficients

[GKP14]

Main results $SL(3), SL(4)$

$$\varphi = \sum_{\mathcal{O}} \varphi_{\mathcal{O}}$$ where $\varphi_{\mathcal{O}}$ vanishes unless $\mathcal{O} \subseteq \mathrm{WF}(\pi)$

Corollary

$\varphi \in \pi_{\min}$ maximally degenerate Whittaker coefficients single root

[GKP14]

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[GKP14]

Main results $SL(3), SL(4)$

$$\varphi = \sum_{\mathcal{O}} \varphi_{\mathcal{O}}$$ where $\varphi_{\mathcal{O}}$ vanishes unless $\mathcal{O} \subseteq \mathrm{WF}(\pi)$

Corollary

$\varphi \in \pi_{\mathrm{min}}$ single root

$\varphi \in \pi_{\mathrm{ntm}}$ at most two commuting roots

[GKP14]

Main results $SL(3), SL(4)$

Fourier coefficients on maximal parabolic subgroups in the minimal representation

   $\pi_{\min}$

[GKP14]

Main results $SL(3), SL(4)$

Fourier coefficients on maximal parabolic subgroups in the minimal representation

$\pi_{\min}$

Theorem

$F_U(\chi_{\min}, \psi; g) = W_N(\chi_{\min}, \psi'; lg)$ with $l \in L(\mathbb{Q})$ depending on ψ

[GKP14]

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$SL(3), SL(4)$

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$F_U(\chi_{\min}, \psi; g) = W_N(\chi_{\min}, \psi'; lg)$ with $l \in L(\mathbb{Q})$ depending on ψ

Maximal parabolic
Fourier coefficient

[GKP14]

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Known Whittaker coefficient

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Maximal parabolic Fourier coefficient

Maximally degenerate

[GKP14]

Example

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$$U = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

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$$F_U(\chi_{\min}, \psi_U; g) = W_N(\chi_{\min}, \psi'_N; lg) \qquad l = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -m & 1 \end{pmatrix}$$

Example

$$U = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \qquad \psi_U\left(\begin{pmatrix} 1 & x_1 & x_2 \\ & 1 & 0 \\ & & 1 \end{pmatrix}\right) = e^{2\pi i (x_1 + m x_2)}$$

$$N = \left\{ \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \right\} \qquad \psi'_N\left(\begin{pmatrix} 1 & x_1 & x_2 \\ & 1 & x_3 \\ & & 1 \end{pmatrix}\right) = e^{2\pi i x_1}$$

$$F_U(\chi_{\min}, \psi_U; g) = W_N(\chi_{\min}, \psi'_N; lg) \qquad l = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -m & 1 \end{pmatrix}$$

$$\int\limits_{U(\mathbb{Q})\backslash U(\mathbb{A})} E(\chi_{\min}; ug)\overline{\psi_U(u)}\, du = \int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} E(\chi_{\min}; nlg)\overline{\psi'_N(n)}\, dn$$

Other groups

[Work in progress with Ahlén, Liu, Kleinschmidt, Persson]

Other groups

$$SL(n)$$

[Work in progress with Ahlén, Liu, Kleinschmidt, Persson]

Other groups

$$SL(n)$$

$$F_U(\chi_{\min}, \psi; g) = W_N(\chi_{\min}, \psi'; lg)$$ with $l \in L(\mathbb{Q})$ depending on ψ

Maximal parabolic
Fourier coefficient

Maximally degenerate

[Work in progress with Ahlén, Liu, Kleinschmidt, Persson]

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$$F_U(\chi_{\min}, \psi; g) = W_N(\chi_{\min}, \psi'; lg)$$ with $l \in L(\mathbb{Q})$ depending on ψ

Maximal parabolic Fourier coefficient

Maximally degenerate

and similar statement for next-to-minimal representation

[Work in progress with Ahlén, Liu, Kleinschmidt, Persson]

Other groups

Conjecture

A similar relations holds for all simple Lie groups

$$F_U(\chi_{\min}, \psi; g) = W_N(\chi_{\min}, \psi'; lg)$$ with $l \in L(\mathbb{Q})$ depending on ψ

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Maximally degenerate

[GKP14]

[Proof in progress with Gourevitch, Kleinschmidt, Persson, Sahi]

Other groups

Conjecture

A similar relations holds for all simple Lie groups

$F_U(\chi_{\min}, \psi; g) = W_N(\chi_{\min}, \psi'; lg)$ with $l \in L(\mathbb{Q})$ depending on ψ

Would allow us to compute non-perturbative effects that capture information about instantons and black holes

[GKP14]

[Proof in progress with Gourevitch, Kleinschmidt, Persson, Sahi]

Outlook

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- Other compactifications leading to automorphic forms on other groups.

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- Simplification of Fourier coefficients with $\chi_{\min}$ for dimensions lower than three. Kac-Moody groups E_9, E_{10}, E_{11}

  [Fleig-Kleinschmidt, Fleig-Kleinschmidt-Persson]

  How to define "small automorphic representations" for Kac-Moody groups? What is the mechanism behind the vanishing properties?

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  How to define "small automorphic representations" for Kac-Moody groups? What is the mechanism behind the vanishing properties?

- $\mathcal{E}_6 D^6 R^4$ requires extended notion of automorphic forms, the development of which will positively bring new exciting insights to both physics and mathematics.

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Thank you!

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